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Large-Order Estimates for Ground-State Energy Perturbation Series

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A general treatment is given, using path integral methods, of obtaining accurate estimates on the rate of growth at large order of the perturbation coefficients for the lowest eigenvalue (ground-state energy) of a large class of anharmonic oscillators. Simple sufficient conditions are given on the potential $V(x)$ so that accurate upper and lower bounds on the perturbation coefficients may be derived. Several examples are given which generalize previous results. Examples from Euclidean quantum field theory are also considered.

KEY WORDS: Perturbation series; asymptotic expansion; ground-state energy; large deviations; instantons.

1. INTRODUCTION

The fact that divergent perturbation series can in many cases be resummed has proven to be a very useful and interesting development for both quantum mechanics and quantum field theory (see Refs. 1–3 and references therein). However, it is not often easy to prove rigorously that a given perturbation series is divergent and therefore requires resummation. Also, central to the use of resummation are accurate estimates on the rate of growth of the perturbation coefficients at large order. There are presently three rigorous techniques for proving divergence and doing large-order estimates: WKB methods,[4,5] Feynman diagram methods,[6–9,12] and path integral methods.[8–13] In this article, we will give a unified treatment of path integral methods, and to a certain extent Feynman diagram methods, as applied to the perturbation series for the lowest eigenvalue (ground-state energy) of a large class of anharmonic oscillators.

This paper is dedicated to the memory of Mark Kac.

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Specifically, we will consider the asymptotic perturbation series as $\lambda \to 0^+$,

$$E(\lambda) \sim \sum_{k=0}^{\infty} a_k \lambda^k \tag{1.1}$$

for the ground-state energy $E(\lambda)$ of the Hamiltonian

$$H(\lambda) = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2 - 1\right) + \lambda V(x) \tag{1.2}$$

$\lambda \geqslant 0$. Accurate upper and lower bounds on the perturbation coefficients a_k that will prove divergence of the series (1.1) and give the correct rate of growth of a_k as $k \to \infty$ will follow from simple sufficient conditions on the potential $V(x)$. In some cases, path integral methods can be used to find very detailed asymptotics of the a_k for large k [see (4.3)]. However, this requires considerably greater effort, and we will not pursue this here. It is possible, though, to do asymptotics of $|a_k|^{1/k}$ relatively easily by path integral methods, and we will compute such asymptotics for some of the examples of Section 4. The asymptotics of $|a_k|^{1/k}$ can be thought of as large-deviation results[36] and as instanton expansions.[3]

Our results are stated in Section 2 (Theorem 2.4 is our main result) and proofs are given in Section 3. The reader may initially wish to skip Section 3 and go on directly to Section 4, where applications are given. Our applications generalize those results already in the literature. Included in Section 4 are applications to Euclidean quantum field theory. Finally, some possible directions for further research on large-order estimates are discussed in Section 5.

In order to give the reader a better idea of our results, we will briefly summarize what we obtain for the applications in Section 4. Our best results are for polynomial potentials. If

$$V(x) = \sum_{n=0}^{2m} \alpha_n x^n, \qquad m = 2, 3, \ldots \tag{1.3}$$

with either $\alpha_n \geqslant 0$, or $(-1)^n \alpha_n \geqslant 0$, $\forall n$, and $\alpha_{2m} = 1$, then our upper and lower bounds on the a_k are accurate enough to prove that

$$\lim_{k \to \infty} \left| \frac{a_k}{[(m-1)k]!} \right|^{1/k} = c_m \exp[-\inf S(\phi) + m] \tag{1.4}$$

where $c_m = (m-1)^{-(m-1)}$ and

$$S(\phi) = \frac{1}{2}\int_{\mathbb{R}} [(\nabla\phi)^2(s) + \phi^2(s)]\, ds - \ln \int_{\mathbb{R}} \phi^{2m}(s)\, ds \tag{1.5}$$

Previous results[4,10,13] considered only the monomial case, $V(x)=x^{2m}$. The appearance of the infimum of $S(\phi)$, with $S(\phi)$ defined as in (1.5), in the asymptotics of (1.4) is what links these results to large deviations[36] and to instanton expansions.[3]

In Section 4.3 we consider the pressure $p(\lambda)$ for a two-dimensional Euclidean quantum field theory with interaction again given by (1.3) and with the same sign restrictions on the α_n. If a_k is now defined by

$$p(\lambda)\sim\sum_{k=0}^{\infty} a_k\lambda^k, \qquad \lambda\to 0^+$$

then our methods are strong enough to prove that (1.4) again holds for a_k but with $S(\phi)$ now defined by

$$S(\phi)=\frac{1}{2}\int_{\mathbb{R}^2}[(\nabla\phi)^2(x)+\phi^2(x)]\,d^2x-\ln\int_{\mathbb{R}^2}\phi^{2m}(x)\,d^2x$$

The previous result[11] only treated the case of $V(x)=x^4$. Viewed as a large-deviation result, this quantum field theory example is very much outside the type of problem covered by the standard methods.[36]

The case that motivated this paper was the exponential interaction

$$V(x)=x^m e^{\alpha x}$$

with, for example, $m=0, 1, 2, 3,\ldots$, $\alpha>0$. For this case, we prove in Section 4.2 that

$$C_1^k k^{mk}\frac{\exp(a\alpha^2k^2)}{k!}\leqslant(-1)^{k+1}a_k\leqslant C_2^k k^{mk/2}\frac{\exp(b\alpha^2k^2)}{k!} \tag{1.6}$$

where all constants are positive. The $m=0$ case of (1.6) was proven in Ref. 12 by using Feynman diagram techniques. It was in trying to extend this earlier result to the $m\neq 0$ case that we developed the general approach leading to Theorem 2.4.

Lastly, we mention a more unusual potential, namely

$$V(x)=x^{2m}[1+\cos(x^{3/2})], \qquad m=0, 1, 2,\ldots$$

While all our other examples are monotone functions for large $|x|$, this potential oscillates with unbounded amplitude as $x\to+\infty$. However, our methods still apply and in Section 4.2 we prove that

$$C_1^k k^{(4m-3/2)k}\frac{\exp(ak^4)}{k!}\leqslant(-1)^{k+1}a_k\leqslant C_2^k k^{mk}\frac{\exp(bk^4)}{k!}$$

again with all constants positive. All of these examples will be discussed in much greater detail in Section 4.

We have not attempted to write our results in the utmost generality, but have stuck to the one-dimensional anharmonic oscillator to keep the paper pedagogically uncomplicated and close to the motivating examples of the theory. There are certainly other operators with lowest eigenvalue perturbation series for which our results should be relevant—multidimensional anharmonic oscillators being the most obvious example. Since our methods depend on path integrals, an immediate necessary restriction for our results to be applicable to an operator $H(\lambda)$ is that the unperturbed operator $H(0)$ have an inverse whose kernel is positive-definite, and therefore suitable to be the covariance of a Gaussian measure.

2. STATEMENT OF RESULTS

Throughout this paper C, C_1, and C_2 will denote positive constants independent of k, whose values many differ from line to line. The operator $H(\lambda)$ of (1.2) acts on $L^2(\mathbb{R})$. The potential $V(x)$ must satisfy the following four conditions (there will be one additional condition):

(a) $V(x)$ is the restriction to the real axis of an entire function.

(b) $V(x)$ is real and bounded below, $\forall x \in \mathbb{R}$.

(c) $\exists$ positive constants C_1, C_2, and $\beta < 2$ such that $|V(z)| \leqslant C_1 \exp(C_2 |z|^\beta)$, $\forall z \in \mathbb{C}$.

(d) If $V(z) = \sum_{n=0}^{\infty} \alpha_n z^n$, $z \in \mathbb{C}$, then either (i) $\alpha_n \geqslant 0$, $\forall n$, or (ii) $(-1)^n \alpha_n \geqslant 0$, $\forall n$.

We will discuss these conditions at the end of this section and in Section 5. For this section, the reader may wish to keep in mind the best known example, $V(x) = x^{2m}$, $m = 2, 3,...$. Next, we give some definitions and results concerning path integrals. It follows from the Feynman–Kac formula (see, e.g., Ref. 9, Theorem 6.3 and Corollary, and p. 183) that

$$E(\lambda) = -\lim_{T \to \infty} \frac{1}{T} \ln Z_X(\lambda)$$

where

$$Z_X(\lambda) = \int e^{-\lambda V(\phi)}\, d\mu_X(\phi)$$

in which

$$V(\phi) = \int_{-T/2}^{T/2} V(\phi(s))\, ds$$

and $d\mu_X(\phi)$ is the mean-zero Gaussian measure with covariance

$$G_X(s, t) = (-\Delta_X + 1)^{-1} (s, t)$$

G_x is the kernel of the integral operator $(-\Delta_x + 1)^{-1}$, where Δ_x is the Laplacian obeying $X = p$ (periodic), D (Dirichlet), 0 (free) boundary conditions on $[-T/2, T/2]$. The covariances are given explicitly by

$$G_0(s, t) = 1/2e^{-|s-t|}$$
$$G_p(s, t) = G_0(s, t) + (1 - e^{-T})^{-1} e^{-T} \cosh(s - t)$$
$$G_D(s, t) = G_0(s, t) - (1 - e^{-2T})^{-1} e^{-T}[\cosh(s - t) - e^{-T} \cosh(s + t)]$$

for $s, t \in [-T/2, T/2]$. We let $u^X_{k,\lambda}(X_1,..., X_k)$ denote the kth Ursell function (Ref. 9, p. 213) with respect to the measure $\exp[-\lambda V(\phi)]\, d\mu_X(\phi)/Z_X(\lambda)$.

Our first proposition gathers together several necessary preliminary results, the proofs of which are mostly straightforward. The main point of this proposition is the representation (2.1) giving the coefficient a_k in terms of an Ursell function, as in Ref. 9, Section 20.

Proposition 2.1. Assume that V satisfies (a)–(c). Then $H(\lambda)$ is a self-adjoint operator with discrete spectrum. The lowest eigenvalue $E(\lambda)$ is nondegenerate and has an asymptotic perturbation series as $\lambda \to 0^+$

$$E(\lambda) \sim \sum_{k=0}^{\infty} a_k \lambda^k$$

with the coefficients a_k having the representation

$$a_k = \frac{(-1)^{k+1}}{k!} \int_{R^{k-1}} d^{k-1}s\, u^0_{k,0}(V(\phi(0)), V(\phi(s_2)),..., V(\phi(s_k))) \tag{2.1}$$

The first of our main results is a bracketing inequality, originally due to Spencer[13] for the case of $V(x) = x^{2m}$, which compares a_k with the finite-T quantities

$$a^X_k(T) \equiv -\frac{1}{Tk!} \frac{d^k}{d\lambda^k}\bigg|_{\lambda=0} \ln Z_X(\lambda)$$
$$= \frac{(-1)^{k+1}}{Tk!} \int_{-T/2}^{T/2} d^k s\, u^X_{k,0}(V(\phi(s_1)),..., V(\phi(s_k))), \qquad X = \mathrm{p, D, 0} \tag{2.2}$$

Theorem 2.2. If V satisfies (a)–(d), then for all $k \geq 1$ and T,

$$(-1)^{k+1} a^D_k(T) \leq (-1)^{k+1} a_k \leq (-1)^{k+1} a^{\mathrm{p}}_k(T) \tag{2.3}$$

Remark. Our proof actually shows the additional inequality

$$(-1)^{k+1} a_k^{\mathrm{D}}(T) \leqslant (-1)^{k+1} a_k^0(T) \leqslant (-1)^{k+1} a_k \tag{2.4}$$

The free boundary condition lower bound on $(-1)^{k+1} a_k$ is used in Ref. 10. However, (2.3) is sufficient for our purposes in this paper. We have mentioned this additional lower bound, since there might be examples where free boundary conditions are technically easier to work with than Dirichlet.

The partition function $Z_X(\lambda)$ also has an asymptotic perturbation series

$$Z_X(\lambda) \sim \sum_{k=0}^{\infty} b_k^X(T)\, \lambda^k$$

with

$$b_k^X(T) = \frac{(-1)^k}{k!} \int V^k(\phi)\, d\mu_X(\phi)$$

We make a final assumption, implicitly on V, concerning the large-k behavior of $b_k^X(T)$.

(e) $\exists$ positive constants k_0, c, and T independent of k and functions $f_{1,X}(k)$ and $f_{2,X}(k)$ such that for $k \geqslant k_0$ and some T,

$$k^k f_{1,X}^k(k) \leqslant \int V^k(\phi)\, d\mu_X(\phi) \leqslant k^k f_{2,X}^k(k), \qquad X = \mathrm{p, D, 0}$$

where $f_{1,X}(k) \to \infty$ as $k \to \infty$, $f_{2,X}^k(k)$ is log convex, and

$$f_{2,X}^j(j)/f_{1,X}^{j-1}(k-1) = O[(j/k)^{cj}]$$

for $k_0 \leqslant j \leqslant [k/2]$.

For some examples, it is useful to have an alternate form of this assumption.

(e′) $\exists$ positive constants k_0, c, and T independent of k and functions $g_{1,X}(k)$ and $g_{2,X}(k)$ such that for $k \geqslant k_0$ and some T,

$$g_{1,X}^k(k) \leqslant \int V^k(\phi)\, d\mu_X(\phi) \leqslant g_{2,X}^k(k), \qquad X = \mathrm{p, D, 0}$$

with $g_{1,X}(k) \geqslant e^{ck}$.

Assumptions (e) and (e′) will be discussed in Section 5.

It turns out that the coefficients $b_k^X(T)$ are much easier to estimate from above and below than the coefficients a_k or $a_k^X(T)$. Our next result reduces large-order estimates of $a_k^X(T)$ to estimates of $b_k^X(T)$.

Theorem 2.3. Assume V satisfies condition (e). Then

$$a_k^X(T) = -(1/T)\, b_k^X(T)\{1 - O[1/f_{1,X}(k)]\} \tag{2.5a}$$

for $k \geqslant k_0$. If V satisfies condition (e′), then

$$a_k^X(T) = -(1/T)\, b_k^X(T)\{1 - O[k/g_{1,X}(k)]\} \tag{2.5b}$$

for $k \geqslant k_0$.

Remark 1. Assumption (e) will be verified for polynomial potentials in Section 4.1 and assumption (e′) will be verified for exponential potentials in Section 4.2.

Remark 2. For V obeying (a)–(d), the bound

$$(-1)^{k+1}\, a_k^X(T) \leqslant \frac{(-1)^k}{T}\, b_k^X(T)$$

holds for all k. This follows from (3.7), the Feynman graph representations (3.8)–(3.10), and the identical representation

$$-\frac{1}{T} b_k^X(T) = \frac{(-1)^{k+1}}{Tk!} \sum_{n_1=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} \alpha_{n_1} \cdots \alpha_{n_k} \sum_{\gamma \in \Gamma_k} \int_{-T/2}^{T/2} d^k s \prod_{l \in \gamma} G_X(s_{l_i}, s_{l_f})$$

where Γ_k is the same as Γ_k^c in (3.8), but with no connectedness requirement. Since the sum over all graphs contains the sum over connected graphs, the above inequality follows, since all terms in the sums are nonnegative. If this inequality is combined with Theorem 2.2, it follows that

$$(-1)^{k+1}\, a_k \leqslant \frac{(-1)^k}{T}\, b_k^{\mathrm{p}}(T)$$

for any V satisfying (a)–(d).

The point to notice at this stage is that, taken together, Theorems 2.2 and 2.3 reduce upper and lower bounds on a_k to upper and lower bounds on $b_k^X(T)$, $X = \mathrm{p}$, D. As we mentioned before, $b_k^X(T)$ is much easier to estimate. In fact, as the examples of Section 4 will show, the techniques needed for $b_k^X(T)$ will be, roughly speaking, Hölders inequality and hypercontractivity for upper bounds and Jensen's inequality for lower bounds.

Our main result is the following theorem.

Theorem 2.4. Assume V satisfies (a)–(e). Then $\exists$ positive constants C_1, C_2 such that for $k \geqslant k_0$,

$$C_1 k^k \frac{f_{1,\mathrm{D}}^k(k)}{k!} \leqslant (-1)^{k+1} a_k \leqslant C_2 k^k \frac{f_{2,\mathrm{p}}^k(k)}{k!}$$

If instead V satisfies (a)–(e′), then for $k \geqslant k_0$,

$$C_1 \frac{g_{1,\mathrm{D}}^k(k)}{k!} \leqslant (-1)^{k+1} a_k \leqslant C_2 \frac{g_{2,\mathrm{p}}^k(k)}{k!}$$

Remark 1. Theorems 2.3 and 2.4 only require condition (e) or (e′) to hold for a single fixed T. However, for the examples of Section 4, condition (e) or (e′) holds for all T with possibly $f_{i,X}$, $g_{i,X}$, $i = 1, 2$, being T-dependent. Also, the proof of Theorem 2.4 only requires the lower bound of condition (e) or (e′) to hold for $X = \mathrm{D}$ and the upper bound to hold for $X = \mathrm{p}$. It is also worth mentioning that $f_{1,\mathrm{D}}$ and $g_{1,\mathrm{D}}$ may be replaced, respectively, with $f_{1,0}$ and $g_{1,0}$, in the lower bounds of Theorem 2.4, as a result of the Remark following Theorem 2.2.

Remark 2. As mentioned in the Introduction, it is also possible to derive exact asymptotics of $|a_k|^{1/k}$ as $k \to \infty$ by path integral methods. We do not attempt to do this in the generality of Theorem 2.4, but instead we will do such asymptotics for specific cases in Section 4.

Remark 3. The exact values of C_1, C_2 are $C_2 = 1/T$, $C_1 = (1/T)(1 - \varepsilon)$ for some $0 < \varepsilon < 1$. The constants C_1, C_2 follow from the $(1/T)\{1 - O[1/f_{1,X}(k)]\}$ term in (2.5a) and the similar term in (2.5b).

It should be emphasized that it is only assumption (e) or (e′) that places a condition on V that guarantees that the series (1.1) is divergent. The bracketing inequality (2.3), which only assumes (a)–(d), can hold even when (1.1) converges, such as when $V(x) = x^2$. In regards to the other conditions, assumption (b) ensures that $H(\lambda)$ is self-adjoint and has a lowest eigenvalue. Assumption (c) guarantees that the asymptotic series (1.1) exists. If V grows too fast, say $V(x) = \exp(x^2)$, then derivatives of $E(\lambda)$ at $\lambda = 0^+$ will fail to exist beyond a certain order. Condition (d) is the crucial assumption for the proof of Theorem 2.2, which should be clear from the discussion following (3.6) and (3.7).

3. PROOFS OF RESULTS

Proof of Proposition 2.1. Since V is real and bounded below, it easily follows that $H(\lambda)$ is self-adjoint, for example, by taking the Friedrichs extension.[14] Alternately, $V \in L^2_{\mathrm{loc}}(\mathbb{R})$, since V is continuous, so that Kato's inequality yields (essential) self-adjointness (Ref. 14, Theorem X.28). Now the term $\frac{1}{2}x^2+\lambda V(x)\to\infty$ as $|x|\to\infty$ and is in $L^1_{\text{loc}}(\mathbb{R})$, so $H(\lambda)$ has only a discrete spectrum with a complete set of eigenfunctions (Ref. 15, Theorem XIII.67). The nondegeneracy of the ground state follows from $\frac{1}{2}x^2+\lambda V(x)$ being positive and in $L^2_{\text{loc}}(\mathbb{R})$ (Ref. 15, Theorem XIII.47).

The proof that $E(\lambda)$ has an asymptotic perturbation series and that (2.1) holds directly follows Simon's proof for the x^4 case (Ref. 9, Section 20). Our assumptions (a)–(c) allow us to replace x^4 with $V(x)$ everywhere in Simon's proof. We sketch the proof for completeness. If we let

$$E_T(\lambda)=-(1/T)\ln Z_X(\lambda)$$

then Lemma 20.1(b) of Ref. 9 shows that existence of an asymptotic perturbation series for $E(\lambda)$ will follow from a bound

$$\left|\frac{d^k}{d\lambda^k}E_T(\lambda)\right|\leqslant C_k,\qquad k\geqslant 0 \tag{3.1}$$

with $0\leqslant\lambda\leqslant 1$, and C_k independent of T. Using the Ursell function representation, as in (2.2), we see that

$$\begin{aligned}\frac{d^k}{d\lambda^k}E_T(\lambda)&=\frac{(-1)^{k+1}}{T}\int_{-T/2}^{T/2}d^ks\,u^0_{k,\lambda}(V(\phi(s_1)),\ldots,V(\phi(s_k)))\\&=\frac{(-1)^{k+1}k!}{T}\int_{-T/2<s_1<\cdots<s_k<T/2}d^ks\,u^0_{k,\lambda}(V(\phi(s_1)),\ldots,V(\phi(s_k)))\end{aligned} \tag{3.2}$$

The bound (3.1) is proven by showing that

$$|u^0_{k,\lambda}(V(\phi(s_1)),\ldots,V(\phi(s_k)))|\leqslant A_k\exp(-B\,|s_j-s_{j-1}|) \tag{3.3}$$

for any $2\leqslant j\leqslant k$. By using

$$|s_k-s_1|\leqslant k\max_{2\leqslant j\leqslant k}|s_j-s_{j-1}|$$

with (3.3) we obtain

$$|u^0_{k,\lambda}(V(\phi(s_1)),\ldots,V(\phi(s_k)))|\leqslant A_k\exp(-B\,|s_k-s_1|/k) \tag{3.4}$$

which, when combined with (3.2), yields (3.1). In order to prove (3.3), Simon next uses Cartier's formula (Ref. 9, p. 131) to write the Ursell function as

$$u^0_{k,\lambda}(V(\phi(s_1)),\ldots,V(\phi(s_k)))=(1/k)\langle\tilde{X}(s_1)\cdots\tilde{X}(s_k)\rangle_\lambda$$

where $\langle \cdot \rangle_\lambda$ is the expectation with respect to

$$\bigotimes_{j=1}^{k} \exp[-\lambda V(\phi_j)]\, d\mu_0(\phi_j)/Z(\lambda)$$

All other notation is as in Ref. 9, p. 214, except that $q_j(s_j)^4$ should be replaced by $V(\phi_j(s_j))$, x_j^4 by $V(x_j)$, and N by $T/2$. The argument leading up to the estimate

$$|\langle \tilde{X}(s_1) \cdots \tilde{X}(s_k) \rangle| \leqslant Z^{-1} e^{-\varepsilon t_{i-1}} \|\tilde{X} e^{-t_{i-2}\hat{L}} \cdots \tilde{\Omega}_0\| \; \|\tilde{X} e^{-t_i \hat{L}} \cdots \tilde{\Omega}_0\| \tag{3.5}$$

remains unchanged, where $\varepsilon \equiv \inf_{0 \leqslant \lambda \leqslant 1} [E_2(\lambda) - E(\lambda)]$ is still strictly positive, since the spectrum of $H(\lambda)$ is discrete and $E(\lambda)$ is nondegenerate. The partition function

$$Z = (\tilde{\Omega}_0, e^{-T\hat{L}} \tilde{\Omega}_0)$$

is decreasing in T, so that $Z \geqslant (\tilde{\Omega}_0, \Omega)^2$. The proof that the norms on the right side of (3.5) are bounded still holds, since this depended on three facts about x^4 that are still true for our $V(x)$. They are that $V(x)$ is bounded below, $\langle |\tilde{X}(\phi(0))|^m \rangle$ is finite for all m, and $|\tilde{X}|^m \exp(-\hat{L})$ is a bounded operator for all m. The latter two assertions hold because of our assumption (c) on the growth of $V(x)$.

In order to prove (2.1), we let $\tilde{a}_k$ equal the right side of (2.1), so that

$$\tilde{a}_k = (-1)^{k+1} \int_{0 < s_2 < \cdots < s_k} d^{k-1}s\, u_{k,0}^0(V(\phi(0)), V(\phi(s_2)), \ldots, V(\phi(s_k)))$$

Similarly,

$$a_k^0(T) = \frac{(-1)^{k+1}}{T} \int_{-T/2 < s_1 < \cdots < s_k < T/2} d^k s\, u_{k,0}^0(V(\phi(s_1)), \ldots, V(\phi(s_k)))$$

Since we know that $a_k = \lim_{T \to \infty} a_k^0(T)$, (2.1) will follow from showing that $a_k^0(T) \to \tilde{a}_k$ as $T \to \infty$. We may write $\tilde{a}_k$ as

$$\tilde{a}_k = \frac{(-1)^{k+1}}{T} \int_{\substack{-T/2 < s_1 < T/2 \\ s_1 < s_2 < \cdots < s_k}} d^k s\, u_{k,0}^0(V(\phi(s_1)), \ldots, V(\phi(s_k)))$$

which gives us

$$|\tilde{a}_k - a_k^0(T)| = \frac{1}{T} \int_{\substack{-T/2 < s_1 < T/2 < s_k \\ s_1 < \cdots < s_k}} d^k s\, u_{k,0}^0(V(\phi(s_1)), \ldots, V(\phi(s_k)))$$

Using (3.4), we obtain

$$|\tilde{a}_k - a_k^0(T)| \leqslant \frac{A_k}{T} \int_{-T/2}^{T/2} ds_1 \int_{s_1}^{s_3} ds_2 \cdots \int_{s_{k-2}}^{s_k} ds_{k-1} \int_{T/2}^{\infty} ds_k \, e^{-B|s_k - s_1|/k}$$

$$\leqslant \frac{A_k}{T} \int_{-T/2}^{T/2} ds_1 \int_{T/2}^{\infty} ds_k \, |s_k - s_1|^{k-2} e^{-B|s_k - s_1|/k}$$

$$= O(1/T)$$

where the last estimate follows as in Ref. 9, p. 216, which proves (2.1). ∎

Proof of Theorem 2.2. If the power series for V,

$$V(z) = \sum_{n=0}^{\infty} \alpha_n z^n$$

is substituted into (2.1) and (2.2), we obtain

$$a_k = \frac{(-1)^{k+1}}{k!} \sum_{n_1=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} \alpha_{n_1} \cdots \alpha_{n_k} \int_{R^{k-1}} d^{k-1}s \, u_{k,0}^0(\phi^{n_1}(0), \phi^{n_2}(s_2), \ldots, \phi^{n_k}(s_k)) \tag{3.6}$$

$$a_k^X(T) = \frac{(-1)^{k+1}}{Tk!} \sum_{n_1=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} \alpha_{n_1} \cdots \alpha_{n_k} \int_{-T/2}^{T/2} d^k s \, u_{k,0}^X(\phi^{n_1}(s_1), \ldots, \phi^{n_k}(s_k)) \tag{3.7}$$

These representations are basic to our proof. The convergence of the above series is a consequence of assumption (c) and will be verified at the end of the proof of this theorem. The importance of assumption (d) is that all nonzero terms in the above series have $\alpha_{n_1} \cdots \alpha_{n_k} \geqslant 0$. This follows for (d)(ii) from the fact that if the number of terms in $\alpha_{n_1} \cdots \alpha_{n_k}$ with n_i odd, $i = 1, \ldots, k$, is odd, then the Ursell function is zero. We will show next in the proof that the integrals of the Ursell functions are also nonnegative, so the sums in (3.6) and (3.7) are entirely of nonnegative terms. This allows us to prove (2.3) by comparing (3.6) and (3.7) term by term.

The point of expanding the V's in their power series is that we are now dealing with Ursell functions of monomials, and such Ursell functions have a very useful expression in terms of Feynman graphs. In order to see this, we define

$$\Gamma_k^c = \{\text{connected graphs} \mid k \text{ vertices, vertex } i \text{ has } n_i \text{ lines attached, } i = 1, \ldots, k\}$$

For each graph γ in Γ_k^c, the total number of lines $n_1 + \cdots + n_k$ must be even or else γ is the empty graph. It will be useful to think of each line in a graph as having a direction, with an initial and final vertex. For line l, s_{l_i} will be the integration variable for the initial vertex and s_{l_f} will be the integration variable for the final vertex. Using Lemma 20.5 of Ref. 9, it is easy to see that

$$\int_{R^{k-1}} d^{k-1}s\, u^0_{k,0}(\phi^{n_1}(0), \phi^{n_2}(s_2),..., \phi^{n_k}(s_k)) = \sum_{\gamma \in \Gamma^c_k} \int_{R^{k-1}} d^{k-1}s \prod_{l\in\gamma} G_0(s_{l_i} - s_{l_f}) \tag{3.8}$$

where γ denotes a graph and the product is over all lines l in γ. Similarly,

$$\frac{1}{T}\int_{-T/2}^{T/2} d^k s\, u^{\mathrm{p}}_{k,0}(\phi^{n_1}(s_1),..., \phi^{n_k}(s_k)) = \int_{-T/2}^{T/2} d^{k-1}s\, u^{\mathrm{p}}_{k,0}(\phi^{n_1}(0), \phi^{n_2}(s_2),..., \phi^{n_k}(s_k))$$

$$= \sum_{\gamma\in\Gamma^c_k} \int_{-T/2}^{T/2} d^{k-1}s \prod_{l\in\gamma} G_p(s_{l_i} - s_{l_f}) \tag{3.9}$$

where for purposes of comparison with (3.8) we have used the periodicity and joint translation invariance in $s_1,..., s_k$ of $u^{\mathrm{p}}_{k,0}$ to cancel the $(1/T)$ factor. We can also do the Dirichlet case to obtain

$$\frac{1}{T}\int_{-T/2}^{T/2} d^k s\, u^{\mathrm{D}}_{k,0}(\phi^{n_1}(s_1),..., \phi^{n_k}(s_k)) = \sum_{\gamma\in\Gamma^c_k} \frac{1}{T}\int_{-T/2}^{T/2} d^k s \prod_{l\in\gamma} G_{\mathrm{D}}(s_{l_i}, s_{l_f}) \tag{3.10}$$

A similar formula would hold for the case $X=0$, but we will not need that case for the proof of Theorem 2.2. Notice that it is an immediate consequence of the Feynman diagram representations (3.8)–(3.10) that the integrals of Ursell functions in (3.6) and (3.7) are nonnegative.

In order to prove the upper bound of (2.3), we substitute the method of images formula

$$G_{\mathrm{p}}(s_{l_i} - s_{l_f}) = \sum_{m\in Z} G_0(s_{l_i} - s_{l_f} + mT)$$

into (3.9) to obtain

$$\int_{-T/2}^{T/2} d^{k-1}s \prod_{l\in\gamma} G_{\mathrm{p}}(s_{l_i} - s_{l_f}) = \sum_{\substack{m_l\in Z\\ l\in\gamma}} \int_{-T/2}^{T/2} d^{k-1}s \prod_{l\in\gamma} G_0(s_{l_i} - s_{l_f} + m_l T) \tag{3.11}$$

We obtain a comparison with (3.8) by writing

$$\int_{R^{k-1}} d^{k-1}s \prod_{l\in\gamma} G_0(s_i - s_j) = \sum_{\substack{n_i\in Z\\ i=1,...,k-1}} \int_{(n_i-1/2)T}^{(n_i+1/2)T} d^{k-1}s \prod_{l\in\gamma} G_0(s_i - s_j)$$

$$= \sum_{\substack{n_i\in Z\\ i=1,...,k-1}} \int_{-T/2}^{T/2} d^{k-1}s \prod_{l\in\gamma} G_0(s_i - s_j + (n_i - n_j)\, T) \tag{3.12}$$

where we are indexing the integration variables by the vertices. Now, all terms in the sum in (3.12) also appear in (3.11), so combining with (3.6)–(3.9) proves that $(-1)^{k+1} a_k \leqslant (-1)^{k+1} a_k^{\mathrm{p}}(T)$. For the lower bound in (2.3), we begin by writing (3.8) as

$$\int_{R^{k-1}} d^{k-1}s\, u_{k,0}^{0}(\phi^{n_1}(0), \phi^{n_2}(s_2),..., \phi^{n_k}(s_k))$$

$$= \lim_{V\to\infty} \sum_{\gamma\in\Gamma_k^c} \frac{1}{V}\int_{-V/2}^{V/2} d^k s \prod_{l\in\gamma} G_0(s_i - s_j) \tag{3.13}$$

where $V=MT$ for T fixed with $M\to\infty$ through positive integers. The integration variables are again being indexed by the vertices. Splitting up the integration as follows, we have

$$\begin{aligned}
&\frac{1}{V}\int_{-V/2}^{V/2} d^k s \prod_{l\in\gamma} G_0(s_i - s_j)\\
&= \frac{1}{MT} \sum_{\substack{n_i = -(M-1)/2\\ i=1,...,k}}^{(M-1)/2} \int_{(n_i - 1/2)T}^{(n_i+1/2)T} d^k s \prod_{l\in\gamma} G_0(s_i - s_j)\\
&= \frac{1}{MT} \sum_{\substack{n_i = -(M-1)/2\\ i=1,...,k}}^{(M-1)/2} \int_{-T/2}^{T/2} d^k s \prod_{l\in\gamma} G_0(s_i - s_j + (n_i - n_j)\,T)\\
&\geqslant \frac{1}{T}\int_{-T/2}^{T/2} d^k s \prod_{l\in\gamma} G_0(s_i - s_j)\\
&\geqslant \frac{1}{T}\int_{-T/2}^{T/2} d^k s \prod_{l\in\gamma} G_D(s_i, s_j)
\end{aligned} \tag{3.14}$$

where in the next to last inequality of (3.14) we have dropped all terms for which $n_i \neq n_j$ and we have used the translation invariance of $G_0(s_i - s_j)$. The last inequality follows from the pointwise inequality $G_0(s_i - s_j) \geqslant G_D(s_i, s_j)$. This finishes the proof that $(-1)^{k+1} a_k^D(T) \leqslant (-1)^{k+1} a_k$. Notice that in the last two inequalities of (3.14) we have actually shown that

$$(-1)^{k+1} a_k^D(T) \leqslant (-1)^{k+1} a_k^0(T) \leqslant (-1)^{k+1} a_k$$

We must still check the convergence of the series in (3.6) and (3.7). We will prove convergence for (3.6), since an almost identical proof works for (3.7). First, if α_n is the nth Taylor coefficient for $V(z)$, then subsituting the bound in (c) into the Cauchy integral formula

$$\alpha_n = \frac{1}{2\pi i}\int_{|z|=r} \frac{V(z)}{z^{n+1}}\, dz$$

and choosing $r = n^{1/\beta}$ gives us

$$|\alpha_n| \leqslant C_1 e^{C_2 n} n^{-n/\beta}$$

This yields

$$|\alpha_{n_1} \cdots \alpha_{n_k}| \leqslant C_1^k \prod_{i=1}^{k} e^{C_2 n_i} n_i^{-n_i/\beta} \tag{3.15}$$

where we take $n_i^{-n_i/\beta} = 1$ if $n_i = 0$.

Next, we estimate the right side of (3.8), exactly as in Ref. 9, pp. 220–221. For each graph $\gamma \in \Gamma_k^c$, we make the change of variables $x_l = s_{l_i} - s_{l_f}$, where $l = 1,..., k-1$ label a choice of $k-1$ lines that connects γ. This leads to

$$\int_{R^{k-1}} d^{k-1}s \prod_{l \in \gamma} G_0(s_{l_i} - s_{l_f}) \leqslant \left[\int_R dx\, G_0(x) \right]^{k-1} \left(\frac{1}{2}\right)^{n_1 + \cdots + n_k - k + 1}$$
$$= C^k \left(\frac{1}{2}\right)^{n_1 + \cdots + n_k}$$

and so

$$\sum_{\gamma \in \Gamma_k^c} \int_{R^{k-1}} d^{k-1}s \prod_{l \in \gamma} G_0(s_{l_i} - s_{l_f}) \leqslant \#(\Gamma_k^c)\, C^k \left(\frac{1}{2}\right)^{n_1 + \cdots + n_k} \tag{3.16}$$

where $\#(\Gamma_k^c)$ is the total number of graphs in Γ_k^c. If we drop the restriction that a graph be connected, we get an upper bound of

$$\#(\Gamma_k^c) \leqslant \frac{(n_1 + \cdots + n_k)!}{[(n_1 + \cdots + n_k)/2]!\, 2^{(n_1 + \cdots + n_k)/2}} \tag{3.17}$$

If we combine together our estimates (3.15)–(3.17), we have

$$|\alpha_{n_1} \cdots \alpha_{n_k}| \int_{R^{k-1}} d^{k-1}s\, u_{k,0}^0(\phi^{n_1}(0), \phi^{n_2}(s_2),..., \phi^{n_k}(s_k))$$
$$\leqslant C_1^k \frac{(n_1 + \cdots + n_k)!}{[(n_1 + \cdots + n_k)/2]!} \prod_{i=1}^{k} e^{C_2 n_i} n_i^{-n_i/\beta} \tag{3.18}$$

Since $(n_1 + n_2)! \geqslant n_1!\, n_2!$, the denominator may be estimated as

$$[(n_1 + \cdots + n_k)/2]! \geqslant \prod_{i=1}^{k} (n_i/2)!$$

The representation

$$n! = \int_0^\infty e^{-t} t^n \, dt$$

allows us to use Hölder's inequality to bound the factorial in the numerator of (3.18) by

$$(n_1 + \cdots + n_k)! \leqslant \prod_{i=1}^{k} [(p_i n_i)!]^{1/p_i}$$

where

$$\sum_{i=1}^{k} p_i^{-1} = 1$$

We can simplify our factorials by using the estimates (Ref. 16, p. 200)

$$e^{7/8} n^{n+1/2} e^{-n} < n! < e n^{n+1/2} e^{-n} \tag{3.19}$$

The lower bound of (3.19) gives us

$$[(n_1 + \cdots + n_k)/2]! \geqslant C^k \prod_{i=1}^{k} [n_i^{n_i/2} n_i^{1/2} (2e)^{-n_i/2}]$$

while

$$(n_1 + \cdots + n_k)! \leqslant C_1 \prod_{i=1}^{k} (n_i^{n_i} n_i^{1/2k} k^{n_i} C_2^{n_i})$$

follows from the upper bound of (3.19) and from choosing $p_i = k$, $\forall i$. Applying these bounds to (3.18) yields

$$\begin{aligned} &|\alpha_{n_1} \cdots \alpha_{n_k}| \int_{R^{k-1}} d^{k-1}s \, u_{k,0}^0(\phi^{n_1}(0), \phi^{n_2}(s_2),..., \phi^{n_k}(s_k)) \\ &\quad \leqslant C_1^k \prod_{i=1}^{k} (n_i^{-n_i(2-\beta)/2\beta} k^{n_i} C_2^{n_i}) \end{aligned} \tag{3.20}$$

which is sufficient to prove convergence of the series in (3.6) since $\beta < 2$. To be specific, write the series as

$$\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} \beta_{n_1,n_2,...,n_k} z_1^{n_1} z_2^{n_2} \cdots z_k^{n_k}$$

where

$$\beta_{n_1,n_2,...,n_k} = \alpha_{n_1} \alpha_{n_2} \cdots \alpha_{n_k} \int_{R^{k-1}} d^{k-1}s \, u_{k,0}^0(\phi^{n_1}(0), \phi^{n_2}(s_2),..., \phi^{n_k}(s_k))$$

and we wish to prove convergence at $z_1 = \cdots = z_k = 1$. The bound (3.20) shows that

$$\sum_{n_1,\ldots,n_k} |\beta_{n_1,\ldots,n_k} z_1^{n_1} \cdots z_k^{n_k}| \leqslant C_1^k \prod_{i=1}^{k} \left(\sum_{n_i} k^{n_i} C_2^{n_i} n_i^{-n_i(2-\beta)/2\beta} |z_i|^{n_i} \right)$$

which is sufficient to prove absolute convergence for all $z_1, z_2, \ldots, z_k$. ∎

Proof of Theorem 2.3. Our proof is a generalization of one we used earlier in Ref. 10, Lemma 2.2. We will drop the notation X and T from $a_k^X(T)$ and $b_k^X(T)$, since the boundary condition dependence and T dependence are irrelevent for our proof. We will also assume that $V(x) \geqslant 0$. This will be justified at the end of the proof. Combining the definition (2.2) with the Taylor series for $\ln(1+x)$ gives us

$$\begin{aligned} a_k &= -\frac{1}{T} \sum_{m=1}^{k} \frac{(-1)^{m-1}}{m} B(k, m) \\ &= -\frac{1}{T} b_k \left[1 + \frac{1}{b_k} \sum_{m=2}^{k} \frac{(-1)^{m-1}}{m} B(k, m) \right] \end{aligned} \tag{3.21}$$

in which

$$B(k, m) = \sum_{\substack{k_1 + \cdots + k_m = k \\ k_i \geqslant 1}} \prod_{i=1}^{m} b_{k_i}$$

Two relations that will be important in our proof are

$$B(k, m) = \sum_{n=1}^{k-m+1} b_n B(k-n, m-1) \tag{3.22}$$

and

$$B(k, 2) = \sum_{k_1=1}^{k-1} b_{k_1} b_{k-k_1} \tag{3.23}$$

We will use (3.23) to show that

$$|B(k, 2)| \leqslant C\, |b_{k-1}| \tag{3.24}$$

and it will then follow inductively from (3.22) that

$$|B(k, m)| \leqslant C^{m-1}\, |b_{k-m+1}| \tag{3.25}$$

for $k \geqslant 2$. Assuming (3.24) for the moment, we will give the rest of the proof of the theorem. If (3.25) holds for $m-1$ and $k \geqslant 2$, then, by (3.22),

$$\begin{aligned}|B(k, m)| &\leqslant \sum_{n=1}^{k-m+1} |b_n|\, C^{m-2}\, |b_{k-n-m+2}| \\ &= C^{m-2}\, |B(k-m+2, 2)| \\ &\leqslant C^{m-1}\, |b_{k-m+1}|\end{aligned}$$

which proves (3.25). The point of (3.25) is that

$$\frac{1}{b_k} B(k, m) = \frac{|B(k, m)|}{|b_k|} \leqslant C^{m-1} \left|\frac{b_{k-m+1}}{b_k}\right|$$

Now, since $V(x) \geqslant 0$,

$$b_k = \frac{(-1)^k}{k!} \|V(\phi)\|_k^k$$

so that

$$\begin{aligned}\frac{|b_{k-m+1}|}{|b_k|} &= \frac{k!}{(k-m+1)!} \frac{\|V(\phi)\|_{k-m+1}^{k-m+1}}{\|V(\phi)\|_k^k} \\ &\leqslant \frac{k!}{(k-m+1)!} \frac{\|V(\phi)\|_k^{k-m+1}}{\|V(\phi)\|_k^k} \\ &= \frac{k!}{(k-m+1)!} \frac{1}{\|V(\phi)\|_k^{m-1}}\end{aligned} \tag{3.26}$$

where we have used Hölder's inequality. Applying the lower bound of assumption (e) to (3.26) yields [we omit the X notation from $f_{i,X}(k)$, $g_{i,X}(k)$]

$$\begin{aligned}\frac{|b_{k-m+1}|}{|b_k|} &\leqslant \frac{k!}{(k-m+1)!} k^{-m+1} C^{m-1} \left[\frac{1}{f_1(k)}\right]^{m-1} \\ &\leqslant C^{m-1} \left[\frac{1}{f_1(k)}\right]^{m-1}\end{aligned}$$

which shows that

$$(1/b_k)\, B(k, m) \leqslant C^{m-1} [1/f_1(k)]^{m-1} \tag{3.27}$$

When (3.27) is combined with (3.21) we obtain

$$a_k = -(1/T)\, b_k\{1 - O[1/f_1(k)]\}$$

so the proof of the theorem is reduced to demonstrating (3.24). If the lower bound of (e′) had been used instead, then (3.27) would be replaced by

$$(1/b_k)\, B(k, m) \leqslant C^{m-1}[k/g_1(k)]^{m-1}$$

which, together with (3.21), proves (2.5b).

In order to prove (3.24), we write $|B(k, 2)|$ as

$$|B(k, 2)| = |b_{k-1}| \left(2\,|b_1| + 2 \sum_{j=2}^{k_0-1} |b_j b_{k-j}/b_{k-1}| + \sum_{j=k_0}^{k-k_0} |b_j b_{k-j}/b_{k-1}| \right) \tag{3.28}$$

The argument leading up to (3.27) then shows that

$$\sum_{j=2}^{k_0-1} |b_j b_{k-j}/b_{k-1}| = O[1/f_1(k-1)] \tag{3.29a}$$

for assumption (e), and that

$$\sum_{j=2}^{k_0-1} |b_j b_{k-j}/b_{k-1}| = O[(k-1)/g_1(k-1)] \tag{3.29b}$$

for assumption (e′). It is only in treating the sum from k_0 to $k-k_0$ that we will use different arguments for (e) and (e′). We will do (e′) first, since this case is easier. Since the integral $\int V^k(\phi)\, d\mu(\phi)$ is log convex in k,

$$\int V^j(\phi)\, d\mu(\phi) \int V^{k-j}(\phi)\, d\mu(\phi) \leqslant \int V^{k_0}(\phi)\, d\mu(\phi) \int V^{k-k_0}(\phi)\, d\mu(\phi)$$

for $k_0 \leqslant j \leqslant k-k_0$. This gives us an upper bound of

$$\sum_{j=k_0}^{k-k_0} \left| \frac{b_j b_{k-j}}{b_{k-1}} \right| \leqslant k \left[\max_{k_0 \leqslant j \leqslant k-k_0} \frac{(k-1)!}{j!\,(k-j)!} \right] \frac{\int V^{k_0}(\phi)\, d\mu(\phi) \int V^{k-k_0}(\phi)\, d\mu(\phi)}{\int V^{k-1}(\phi)\, d\mu(\phi)} \tag{3.30}$$

If we apply the estimates

$$\frac{\int V^{k-k_0}(\phi)\, d\mu(\phi)}{\int V^{k-1}(\phi)\, d\mu(\phi)} \leqslant \frac{1}{\|V(\phi)\|_{k-1}^{k_0-1}} \leqslant \frac{1}{g_1^{k_0-1}(k-1)}$$

and

$$\max_{k_0 \leqslant j \leqslant k - k_0} \frac{(k-1)!}{j!\,(k-j)!} \leqslant C k^{-3/2} 2^k$$

to (3.30), we find that

$$\begin{aligned} \sum_{j=k_0}^{k-k_0} |b_j b_{k-j} / b_{k-1}| &\leqslant C 2^k / g_1^{k_0 - 1}(k-1) \\ &\leqslant C 2^k e^{-(k_0 - 1)ck} \\ &= O(e^{-C_1 k}) \end{aligned} \tag{3.31}$$

where the next to last line uses $g_1(k) \geqslant e^{ck}$. Combining (3.28), (3.29b), and (3.31) proves (3.24) for (e′).

For the case of (e), we first pick a $0 < \theta < 1$ and consider $k_0 \leqslant j \leqslant [k^\theta]$. The argument leading to (3.27) shows that

$$\begin{aligned} \left| \frac{b_j b_{k-j}}{b_{k-1}} \right| &\leqslant \frac{\int V^j(\phi)\, d\mu(\phi)}{j!} \frac{1}{f_1^{j-1}(k-1)} \leqslant C \frac{j^j f_2^j(j)}{j^j j^{1/2} e^{-j}} \frac{1}{f_1^{j-1}(k-1)} \\ &\leqslant C e^j \frac{f_2^j(j)}{f_1^{j-1}(k-1)} = O[e^j (j/k)^{cj}] = O(k^{-ck_0}) \end{aligned}$$

for $k_0 \leqslant j \leqslant [k^\theta]$, where the next to last line uses assumption (e). This yields

$$\sum_{j=k_0}^{[k^\theta]} |b_j b_{k-j} / b_{k-1}| = O(k^{-ck_0 + 1}) \tag{3.32}$$

Next, for $[k^\theta] \leqslant j \leqslant [k/2]$ we have

$$\begin{aligned} \left| \frac{b_j b_{k-j}}{b_{k-1}} \right| &\leqslant C \frac{(k-1)^{k-1/2}}{j^j (k-j)^{k-j}} \frac{j^j f_2^j(j) (k-j)^{k-j} f_2^{k-j}(k-j)}{(k-1)^{k-1} f_1^{k-1}(k-1)} \\ &\leqslant C k^{1/2} \frac{f_2^{[k^\theta]}([k^\theta])\, f_2^{k-[k^\theta]}(k-[k^\theta])}{f_1^{k-1}(k-1)} \\ &= \frac{C k^{1/2}}{f_1(k-1)} \frac{f_2^{[k^\theta]}([k^\theta])}{f_1^{[k^\theta]-1}(k-1)} \frac{f_2^{k-[k^\theta]}(k-[k^\theta])}{f_1^{k-[k^\theta]-1}(k-1)} \\ &= O\left[k^{1/2} \left(\frac{k^\theta}{k} \right)^{ck^\theta} \left(\frac{k-k^\theta}{k} \right)^{c(k-k^\theta)} \right] \\ &= O\{\exp[-c(1-\theta)\, k^\theta \ln k] \exp(ck^{2\theta - 1})\} \\ &= O\{\exp[-c(1-\theta)\, k^\theta \ln k]\} \end{aligned}$$

since $2\theta - 1 < \theta$ for $\theta < 1$. The second line above uses the log convexity of $f_2^k(k)$. We obtain

$$\sum_{j=[k^\theta]}^{[k/2]} |b_j b_{k-j}/b_{k-1}| = O[\exp(-Ck^\theta \ln k)] \tag{3.33}$$

Since

$$\sum_{j=k_0}^{k-k_0} |b_j b_{k-j}/b_{k-1}| \leqslant 2 \sum_{j=k_0}^{[k/2]} |b_j b_{k-j}/b_{k-1}|$$

(3.32) and (3.33) prove (3.24) for (e) and this finishes the proof of the theorem, except for justifying the assumption $V(x) \geqslant 0$. It will suffice to check this for condition (e). Now, the point of assuming $V(x) \geqslant 0$ was that

$$\int V^k(\phi)\, d\mu(\phi) = \| V(\phi)\|_k^k$$

This is certainly true when k is even, so we will assume k odd and prove that

$$\frac{\int |V(\phi)|^k\, d\mu(\phi)}{\int V^k(\phi)\, d\mu(\phi)} = 1 + O\left[\frac{b^k}{k^k f_1^k(k)}\right] \tag{3.34}$$

where b is a positive constant such that $V(x) \geqslant -b$ [assumption (b)]. To do this, let

$$I = \{\phi \mid V(\phi) > 0\}$$

so that we can consider

$$\frac{\int |V(\phi)|^k\, d\mu(\phi)}{\int V^k(\phi)\, d\mu(\phi)} = \frac{\int \chi_I(\phi)\, V^k(\phi)\, d\mu(\phi) - \int \chi_{\tilde{I}}(\phi)\, V^k(\phi)\, d\mu(\phi)}{\int \chi_I(\phi)\, V^k(\phi)\, d\mu(\phi) + \int \chi_{\tilde{I}}(\phi)\, V^k(\phi)\, d\mu(\phi)} \tag{3.35}$$

The lower bound of assumption (e) gives us

$$\begin{aligned} k^k f_1^k(k) &\leqslant \int V^k(\phi)\, d\mu(\phi) \\ &= \int \chi_I(\phi)\, V^k(\phi)\, d\mu(\phi) + \int \chi_{\tilde{I}}(\phi)\, V^k(\phi)\, d\mu(\phi) \\ &\leqslant \int \chi_I(\phi)\, V^k(\phi)\, d\mu(\phi) \end{aligned}$$

since k is odd. If this lower bound is combined with

$$\left| \int \chi_{\tilde{I}}(\phi)\, V^k(\phi)\, d\mu(\phi) \right| \leqslant b^k$$

then (3.34) follows easily from (3.35). ∎

Proof of Theorem 2.4. The theorem is an immediate consequence of combining (2.3), (2.5a), or (2.5b), and the upper and lower bounds of assumption (e) or (e′). ∎

4. APPLICATIONS

Our applications are natural generalizations of those large-order results already known for anharmonic oscillators. Section 4.1 treats polynomial potentials, while Section 4.2 deals with exponential and mixed exponential–polynomial potentials. In Section 4.2 we also treat an oscillatory potential. Section 4.3 is about two-dimensional Euclidean quantum fields with polynomial interactions. While field theory may seem like a very different application from Sections 4.1 and 4.2, the results are almost identical with those in Section 4.1—a reflection of the fact that anharmonic oscillators are really one-dimensional quantum fields. We should mention that quantum fields are one of the main applications of large-order perturbation theory and of resummation of divergent series.(3,17,19,20)

4.1. Polynomial Potentials

We assume that

$$V(x) = \sum_{n=0}^{2m} \alpha_n x^n, \qquad m = 2, 3,...$$

with α_n obeying either

(i) $\alpha_n \geqslant 0, \forall n$

or

(ii) $(-1)^n \alpha_n \geqslant 0, \quad \forall n$

For simplicity, we assume $\alpha_{2m} = 1$, since this coefficient can be absorbed in the coupling constant λ. For this example, we will also compute the asymptotics of $|a_k|^{1/k}$, so some additional definitions are needed. We will use the following functionals:

$$S(\phi) = \frac{1}{2} \int_R [(\nabla\phi)^2(s) + \phi^2(s)]\, ds - \ln \int_R \phi^{2m}(s)\, ds$$

and

$$S_X(\phi)=\frac{1}{2}\int_{-T/2}^{T/2}[(\nabla_X\phi)^2(s)+\phi^2(s)]\,ds$$

$$-\ln\int_{-T/2}^{T/2}\phi^{2m}(s)\,ds, \qquad X=\mathrm{p},\mathrm{D}$$

in which ∇_X is the gradient obeying X boundary conditions on $[-T/2, T/2]$. The functional $S(\phi)$ is defined on the Sobolev space $W^{1,2}(\mathbb{R})$, which is the completion of $C_0^\infty(\mathbb{R})$ in the norm

$$\|\phi\|_{1,2}^2=\int_R[(\nabla\phi)^2(s)+\phi^2(s)]\,ds$$

The functionals $S_X(\phi)$ are defined on similar Sobolev spaces on $[-T/2, T/2]$ with norms given by

$$\int_{-T/2}^{T/2}[(\nabla_X\phi)^2(s)+\phi^2(s)]\,ds$$

As in Refs. 10 and 11, the functionals $S(\phi)$ and $S_X(\phi)$ are bounded below and attain their infimums.

We state our two results concerning large-order estimates for this example together in the following theorem.

Theorem 4.1. Assume that $V(x)$ is a polynomial obeying either (i) or (ii). Then there exist positive constants k_0, C_1, C_2, D_1, D_2, a, b, γ, τ, with γ, $\tau<1$, such that for all $k\geqslant k_0$,

$$C_1\exp(ak^\gamma)\,D_1^k\frac{k^{mk}}{k!}\leqslant(-1)^{k+1}a_k\leqslant C_2\exp(bk^\tau)\,D_2^k\frac{k^{mk}}{k!} \tag{4.1}$$

Furthermore,

$$\lim_{k\to\infty}\left|\frac{a_k}{[(m-1)k]!}\right|^{1/k}=c_m\exp[-\inf S(\phi)+m] \tag{4.2}$$

where the infimum is taken over all $\phi\in W^{1,2}(\mathbb{R})$, and $c_m=(m-1)^{-(m-1)}$.

Remark 1. The exact values of D_1 and D_2 are

$$D_1=\exp[-\inf S_{\mathrm{D}}(\phi)], \qquad D_2=\exp[-\inf S_{\mathrm{p}}(\phi)]$$

Remark 2. In light of Eqs. (4.6) and (4.18) in the proof, it might seen more natural to state (4.2) as

$$\lim_{k\to\infty}\left|\frac{a_k}{k^{mk}/k!}\right|^{1/k}=\exp[-\inf S(\phi)]$$

However, the right side of (4.2) is actually the reciprocal of the radius of convergence of the Borel transform $B(t)$ for the ground-state energy $E(\lambda)$, where

$$B(t) = \sum_{k=0}^{\infty} \frac{a_k}{[(m-1)k]!} t^k$$

Remark 3. The functionals $S(\phi)$ and $S_X(\phi)$ represent classical actions, and (4.2) is an example of an instanton expansion. That is, (4.2) follows from a Laplace-type asymptotic expansion of the functional integrals appearing in $b_k^X(T)$ about the minima of $S_X(\phi)$. Equation (4.2) can also be viewed as a large-deviation result (compare with Ref. 36, Theorems 2.2 and 5.1).

Before giving the proof of Theorem 4.1, there are several points about polynomial potentials we wish to discuss. The first is that extremely detailed asymptotics can be done in this case. For the case of $V(x) = x^4$, Ref. 10 shows that as $k \to \infty$

$$a_k \sim ab^k k^c \frac{k^{2k}}{k!} \left[1 + \frac{d}{k} + O(k^{-2(1-\beta)}) \right] \tag{4.3}$$

with $0 < \beta < 1/4$ and a, b, c, d explicitly known constants (ideally, we should have $\beta = 0$). The methods of Ref. 10 would work to prove an asymptotic expansion of the form of (4.3) for the general polynomial potential considered in this section—although Ref. 10 is long enough as is, just for the case of x^4! We should also mention that the first rigorous proof of (4.3) for $V(x) = x^4$, obtaining a, b, and c, but not d, was done by WKB methods.[4] As a comparison, WKB methods have the advantage of working also for the higher eigenvalues, while path integral methods apparently do not. As a disadvantage, WKB does not seem to extend to field theories, while path integral methods do (Refs. 8, 11, and Section 4.3).

The results of Theorem 4.1 show that only the x^{2m} term of $V(x)$ matters for the large-order behavior of $|a_k|^{1/k}$. We expect this to be true for any polynomial potential with highest degree term x^{2m}. Our restriction on the signs of the coefficients of the polynomials given by condition (d) of Section 2 is a technical condition necessary for our present proof of Theorem 2.2 and should not really be needed to prove Theorem 4.1. The results in the physics literature[2] imply that for asymptotic expansions of the form of (4.3), lower degree terms in V can only affect the constants a and d and the coefficients of higher order corrections in powers of $1/k$, except for an $\alpha_{2m-1} x^{2m-1}$ term, which will give a contribution of $\exp(c\alpha_{2m-1} k^{1/2})$. The situation is very different, however, for potentials that are nonlinear in the coupling constant λ. Herbst and Simon[21] have some simple examples of potentials $V(x, \lambda)$, polynomial in x and λ, for which lower degree terms in x very drastically affect the perturbation coefficients (see Section 5).

The last point we wish to discuss is the necessity of condition (d). This is an easy to verify sufficient condition for the validity of the bracketing inequality (2.3). While it is crucial for our present proof of (2.3), it is certainly not necessary, as the following simple polynomial examples show. If we take $V(x)=x^4-x^2$, then a straightforward calculation shows that $a_1^{\mathrm{D}}(T)\leqslant a_1\leqslant a_1^{\mathrm{P}}(T)$, and that at least for T sufficiently large,

$$-a_2^{\mathrm{D}}(T)\leqslant -a_2\leqslant -a_2^{\mathrm{P}}(T)$$

so (d) is not a necessary condition. A more striking example is given by taking V to be Wick-ordered x^4,

$$V(x)=:x^4:\equiv x^4-6G_0(0)\,x^2+3G_0(0)^2=x^4-3x^2+\frac{3}{4}$$

We will show in Section 4.3 that the bracketing inequality (2.3) holds for all $k\geqslant 1$ and T for this choice of V. A more detailed treatment of this example will be given in Section 4.3, since the quantum field theory examples naturally require Wick-ordering.

Proof of Theorem 4.1. For the proof of (4.1), we need only verify conditions (a)–(e) with

$$f_{1,X}^k(k)=C_1\exp(ak^\gamma)\,D_X^k k^{(m-1)k}$$
$$f_{2,X}^k(k)=C_2\exp(bk^\tau)\,D_X^k k^{(m-1)k}$$

where $D_X=\exp[-\inf S_X(\phi)]$, and (4.1) is then a consequence of Theorem 2.4. Note that $D_1=D_{\mathrm{D}}$ and $D_2=D_{\mathrm{P}}$. For our polynomial V, conditions (a)–(d) are immediate. It is also easy to check that $f_{2,X}^k(k)$ is log convex and that the estimate on $f_{2,X}^j(j)/f_{1,X}^{j-1}(k-1)$ is true. Therefore, we need to verify the upper and lower bounds of (e). Also, sharp upper and lower bounds in (e) as $k\to\infty$ are what we will need for the proof of (4.2). First we introduce some definitions. Let

$$V_n(\phi)=\int_{-T/2}^{T/2}\phi^n(s)\,ds$$

so that

$$V(\phi)=\sum_{n=0}^{2m}\alpha_n V_n(\phi)$$

and let ϕ^c denote the function that minimizes $S_X(\phi)$, $X = \mathrm{p}, \mathrm{D}$. Of course, ϕ^c is different for the different boundary conditions, but this will not matter in our proof. Also, ϕ^c is not unique when $X = \mathrm{p}$, since $S_{\mathrm{p}}(\phi)$ is translation-invariant. Again, this will not matter for our proof. We verify the lower bound first.

Lower Bound. Our proof is a modified version of our proof of Theorem 1.2 in Ref. 11. We start with the case of even k, since this is easier. Therefore, let $k = 2j$, $j = 1, 2, 3,\ldots$, and consider the following, where $A_X = -\Delta_X + 1$,

$$\int V^k(\phi)\, d\mu_X(\phi) = \exp(-\tfrac{1}{2}k\, \langle \phi^c, A_X\phi^c\rangle) \times \int V^k(\phi + \sqrt{k}\, \phi^c) \exp(-\sqrt{k}\, \langle \phi, A_X\phi^c\rangle)\, d\mu_X(\phi) \tag{4.4}$$

where we have translated $\phi \to \phi + \sqrt{k}\, \phi^c$. Looking at just the integral on the right side of (4.4) gives us

$$\begin{aligned}
&\int V^k(\phi + \sqrt{k}\, \phi^c) \exp(-\sqrt{k}\, \langle \phi, A_X\phi^c\rangle)\, d\mu_X(\phi) \\
&= \int \{V^2(\phi + \sqrt{k}\, \phi^c) \exp[-(2/\sqrt{k})\langle \phi, A_X\phi^c\rangle]\}^j\, du_X(\phi) \\
&\geqslant \left\{\int V^2(\phi + \sqrt{k}\, \phi^c) \exp[-(2/\sqrt{k})\langle \phi, A_X\phi^c\rangle]\, d\mu_X(\phi)\right\}^j \\
&= k^{mk} V^k_{2m}(\phi^c)[1 + O(k^{-1/2})]^{k/2}
\end{aligned} \tag{4.5}$$

where the third line follows from Jensen's inequality. Combining (4.4) and (4.5) shows that

$$\int V^k(\phi)\, d\mu_X(\phi) \geqslant k^{mk}\{\exp[-kS_X(\phi^c)]\}[1 + O(k^{-1/2})]^{k/2} \tag{4.6}$$

which verifies the lower bound of condition (e) for k even. We also obtain from (4.6) that

$$\lim_{k \to \infty} \left|\frac{(-1)^k\, b_k^X(T)}{[(m-1)\,k]!}\right|^{1/k} \geqslant c_m \exp[-S_X(\phi^c) + m] \tag{4.7}$$

again for k even. Inequality (4.7) will be used in the proof of (4.2) and will soon be shown to also hold for k odd.

For the case of k odd, our Jensen's inequality argument does not directly apply, since $V(\phi)$ may be negative. However, we will show that those ϕ for which $V(\phi + \sqrt{k}\,\phi^c) < 0$ are negligible for large k. To do this, define

$$A = \{\phi \mid |V(\phi) - V_{2m}(\phi)| < k^{m-\beta}\}$$
$$B = \{\phi \mid V_{2m}(\phi + \sqrt{k}\,\phi^c) \geqslant k^{m-\beta}\}$$

The following lemma gives us the necessary estimates.

Lemma 4.2. Choose $0 < \beta < 1/2$. Then for k sufficiently large,

$$\mu_X\{\phi \mid \phi \in A^{\sim}\} \leqslant \exp(-Ck^{1+\eta}) \tag{4.8}$$
$$\mu_X\{\phi \mid \phi \in B^{\sim}\} \leqslant \exp(-Ck^{1/2m}) \tag{4.9}$$

where $\eta = (1-2\beta)/(2m-1)$.

Proof of Lemma 4.2. Inequality (4.8) is proven by using

$$\begin{aligned}
\mu_X\{\phi \in A^{\sim}\} &\leqslant k^{-(m-\beta)p} \int |V(\phi) - V_{2m}(\phi)|^p \, d\mu_X(\phi) \\
&\leqslant k^{-(m-\beta)p}(p-1)^{(2m-1)p/2} \, \|V(\phi) - V_{2m}(\phi)\|_2^p \\
&\leqslant k^{-(m-\beta)p} p^{(2m-1)p/2} C^p \\
&\leqslant \exp(-Ck^{1+\eta})
\end{aligned}$$

where the second inequality follows from hypercontractivity[27] and the last inequality from minimizing on p.

For (4.9), write

$$\begin{aligned}
\mu_X\{\phi \mid \phi \in B^{\sim}\} &= \mu_X\{\phi \mid V_{2m}(\phi/\sqrt{k} + \phi^c) < k^{-\beta}\} \\
&= \mu_X\{\phi \mid V_{2m}(\phi/\sqrt{k} + \phi^c) - V_{2m}(\phi^c) < k^{-\beta} - V_{2m}(\phi^c)\} \\
&= \mu_X\{\phi \mid |V_{2m}(\phi^c) - V_{2m}(\phi/\sqrt{k} + \phi^c)| > V_{2m}(\phi^c) - k^{-\beta}\}
\end{aligned} \tag{4.10}$$

in which we assume that k is large enough so that $V_{2m}(\phi^c) > k^{-\beta}$. The last expression in (4.10) may now be estimated exactly as before, using hypercontractivity, the estimate

$$\|V_{2m}(\phi^c) - V_{2m}(\phi/\sqrt{k} + \phi^c)\|_2 = O(k^{-1/2})$$

and minimizing on p to obtain (4.9). ∎

Returning to the proof of the lower bound for k odd, we write

$$\int V^k(\phi)\, d\mu_X(\phi) = \int \chi_A(\phi)\, V^k(\phi)\, d\mu_X(\phi) + \int \chi_{A^{\sim}}(\phi)\, V^k(\phi)\, d\mu_X(\phi) \tag{4.11}$$

where χ_A is the characteristic function of the set A. We estimate the second integral on the right side of (4.11) as

$$\left|\int \chi_{A^\sim}(\phi)\, V^k(\phi)\, d\mu_X(\phi)\right|$$
$$\leqslant \mu_X\{\phi \mid \phi \in A^\sim\}^{1/2}\left[\int V^{2k}(\phi)\, d\mu_X(\phi)\right]^{1/2}$$
$$\leqslant \mu_X\{\phi \mid \phi \in A^\sim\}^{1/2}\,(2k-1)^{mk}\,\|V(\phi)\|_2^k$$
$$= O[\exp(-Ck^{1+\eta})]$$

again using hypercontractivity. Translating $\phi \to \phi + \sqrt{k}\,\phi^c$ in the remaining integral in (4.11) yields

$$\int \chi_{A^\sim}(\phi)\, V^k(\phi)\, d\mu_X(\phi)$$
$$= \exp\left(-\frac{k}{2}\langle \phi^c, A_X\phi^c\rangle\right)$$
$$\times \int \chi_{A'}(\phi)\, V^k(\phi + \sqrt{k}\,\phi^c)\exp(-\sqrt{k}\,\langle \phi, A_X\phi^c\rangle)\, d\mu_X(\phi) \qquad (4.12)$$

in which $\chi_{A'}(\phi) = \chi_A(\phi + \sqrt{k}\,\phi^c)$. Since k is odd, the integral on the right side of (4.12) can now be estimated from below as

$$\int \chi_{A'}(\phi)\, V^k(\phi + \sqrt{k}\,\phi^c)\exp(-\sqrt{k}\,\langle \phi, A_X\phi^c\rangle)\, d\mu_X(\phi)$$
$$\geqslant \int \chi_{A'}(\phi)[V_{2m}(\phi + \sqrt{k}\,\phi^c) - k^{m-\beta}]^k \exp(-\sqrt{k}\,\langle \phi, A_X\phi^c\rangle)\, d\mu_X(\phi) \qquad (4.13)$$

The next step is to split up the integral on the right side of (4.13) by inserting the characteristic functions of B and $B^\sim$. First, we consider

$$\left|\int \chi_{A'\cap B^\sim}(\phi)\,[V_{2m}(\phi + \sqrt{k}\,\phi^c) - k^{m-\beta}]^k \exp(-\sqrt{k}\,\langle \phi, A_X\phi^c\rangle)\, d\mu_X(\phi)\right|$$
$$= \int \chi_{A'\cap B^\sim}(\phi)[k^{m-\beta} - V_{2m}(\phi + \sqrt{k}\,\phi^c)]^k$$
$$\times \exp(-\sqrt{k}\,\langle \phi, A_X\phi^c\rangle)\, d\mu_X(\phi)$$
$$\leqslant k^{(m-\beta)k}\int \chi_{A'\cap B^\sim}(\phi)\exp(-\sqrt{k}\,\langle \phi, A_X\phi^c\rangle)\, d\mu_X(\phi)$$
$$= O(k^{(m-\beta)k}) \qquad (4.14)$$

We will see that this term is smaller by $O(k^{-\beta k})$ than the contribution of the integral containing B. On the set B,

$$\int \chi_{A' \cap B}(\phi)\,[V_{2m}(\phi + \sqrt{k}\,\phi^c) - k^{m-\beta}]^k \exp(-\sqrt{k}\,\langle \phi, A_X \phi^c \rangle)\,d\mu_X(\phi)$$

$$\geqslant \Big[\int \chi_{A' \cap B}(\phi)(V_{2m}(\phi + \sqrt{k}\,\phi^c) - k^{m-\beta})$$

$$\times \exp(-k^{-1/2}\langle \phi, A_X \phi^c \rangle)\,d\mu_X(\phi)\Big]^k \mu_X\{\phi \mid \phi \in A' \cap B\}^{1-k}$$

$$= k^{mk} V_{2m}(\phi^c)[1 + O(k^{-\beta})]^k \{1 + O[\exp(-Ck^{1/2m})]\}^{1-k} \qquad (4.15)$$

The second line follows from Jensen's inequality, and the last line uses (4.9). Combining together (4.11)–(4.15) gives

$$\int V^k(\phi)\,d\mu_X(\phi) \geqslant k^{mk}\{\exp[-kS_X(\phi^c)]\}[1 + O(k^{-\beta})]^k \qquad (4.16)$$

which together with (4.6) completes our proof that the lower bound of condition (e) is true. It also follows from (4.16) that (4.7) holds for k odd.

Upper Bound. For the case of polynomial $V(x)$, the first step in deriving the upper bound of condition (e) is to use hypercontractivity. In particular,

$$\left|\int V^k(\phi)\,d\mu_X(\phi)\right|^{1/k} \leqslant \|V(\phi)\|_k$$

$$\leqslant \|V_{2m}(\phi)\|_k + \sum_{n=0}^{2m-1} |\alpha_n|\ \|V_n(\phi)\|_k$$

$$\leqslant \|V_{2m}(\phi)\|_k + \sum_{n=0}^{2m-1} |\alpha_n|\ k^{n/2}\,\|V_n(\phi)\|_2$$

$$= k^m \left\{\|V_{2m}(\phi/\sqrt{k})\|_k + \sum_{n=0}^{2m-1} |\alpha_n|\ k^{(n-2m)/2}\,\|V_n(\phi)\|_2\right\}$$

so that

$$\int V^k(\phi)\,d\mu_X(\phi) \leqslant k^{mk}\,\|V_{2m}(\phi/\sqrt{k})\|_k^k\,[1 + O(k^{-1/2})]^k \qquad (4.17)$$

We will now analyze the $\|V_{2m}(\phi/\sqrt{k})\|_k$ term for large k to find the required upper bound. This can be done by using the lattice approximation with k-dependent lattice spacing constant δ, exactly as in Ref. 11. We sketch the proof. For details of the lattice approximation see Refs. 26, 27, 30, and 31. The interval $[-T/2, T/2]$ is discretized, with lattice spacing $\delta = T/n$, where $n = 2[k^{\varepsilon}] + 1$ is the number of lattice sites and $\varepsilon < 1$. We will use the finite-dimensional Gaussian measure

$$d\mu_{X,\delta}(q) = \exp\left(-\frac{1}{2}\langle q, A_X^{\delta} q\rangle\right) d^n q\, N_X$$

where N_X is a normalization constant. The covariance is

$$(A_X^{\delta})^{-1} \equiv (-\Delta_X^{\delta} + 1)^{-1}$$

in which Δ_X^{δ} is the finite-difference Laplacian on $[-T/2, T/2]$ with $X = \mathrm{p}$, D boundary conditions (see Ref. 31). We also define

$$V_{2m,\delta}(q) = \delta \sum_{i=-(n-1)/2}^{(n-1)/2} q_i^{2m}$$

and

$$S_X^{\delta}(q) = \frac{1}{2}\langle q, A_X^{\delta} q\rangle - \ln V_{2m,\delta}(q)$$

We recall that the lattice approximation can be defined by the continuum theory as $q_i = \phi(f_{i,\delta}^X)$, where the function $f_{i,\delta}^X$ is given in Ref. 31, Section IX.1.

An upper bound on $\|V_{2m}(\phi/\sqrt{k})\|_k$ now follows from

$$\begin{aligned} &|\ \|V_{2m}(\phi/\sqrt{k})\|_k - \|V_{2m,\delta}(\phi/\sqrt{k})\|_k| \\ &\quad \leqslant \|V_{2m}(\phi/\sqrt{k}) - V_{2m,\delta}(\phi/\sqrt{k})\|_k \\ &\quad \leqslant (k-1)^m \|V_{2m}(\phi/\sqrt{k}) - V_{2m,\delta}(\phi/\sqrt{k})\|_2 \\ &\quad = O(\delta^{\theta}) = O(k^{-\varepsilon\theta}) \end{aligned}$$

for some $\theta > 0$, and so

$$\|V_{2m}(\phi/\sqrt{k})\|_k \leqslant \|V_{2m,\delta}(\phi/\sqrt{k})\|_k + O(k^{-\varepsilon\theta})$$

Therefore, we have

$$\begin{aligned} \|V_{2m}(\phi/\sqrt{k})\|_k^k &\leqslant [\|V_{2m,\delta}(\phi/\sqrt{k})\|_k + O(k^{-\varepsilon\theta})]^k \\ &= \|V_{2m,\delta}(\phi/\sqrt{k})\|_k^k\, [1 + O(k^{-\varepsilon\theta})]^k \end{aligned}$$

where the last line uses that our lower bound argument easily shows that $\| V_{2m,\delta}(\phi/\sqrt{k})\|_k$ is bounded below by a constant. Our upper bound is now reduced to estimating a finite-dimensional integral, since

$$\begin{aligned}
\| V_{2m,\delta}(\phi/\sqrt{k})\|_k^k &= \int V_{2m,\delta}^k(\phi/\sqrt{k})\, d\mu_X(\phi) \\
&= \int V_{2m,\delta}^k(q/\sqrt{k})\, d\mu_{X,\delta}(q) \\
&= \int \exp[-kS_X^\delta(q/\sqrt{k})]\, d^nq\, N_X \\
&= k^{n/2} \int \exp[-kS_X^\delta(q)]\, d^nq\, N_X
\end{aligned}$$

The last integral may be estimated as

$$\begin{aligned}
&\int \exp[-kS_X^\delta(q)]\, d^nq\, N_X \\
&\quad \leqslant \exp[-kS_X^\delta(q^c)] \exp[S_X^\delta(q^c)] \int \exp[-S_X^\delta(q)]\, d^nq\, N_X \qquad (4.18)
\end{aligned}$$

for $k \geqslant 1$. We will write $\exp[-kS_X^\delta(q^c)]$ as

$$\exp[-kS_X^\delta(q^c)] = \exp[-kS_X(\phi^c)] \exp\{-k[S_X^\delta(q^c) - S_X(\phi^c)]\}$$

and now show that

$$|S_X^\delta(q^c) - S_X(\phi^c)| = O(\delta) = O(k^{-\varepsilon})$$

Since ϕ^c is a critical point for $S_X(\phi)$, it is a solution of

$$(-\Delta_X + 1)\, \phi^c(x) = 2m(\phi^c)^{2m-1}(x) \Big/ \int_{-T/2}^{T/2} ds\, (\phi^c)^{2m}(s)$$

and the finite-difference version of this equation holds for q^c. It follows from the critical point equations that

$$\langle \phi^c, (-\Delta_X + 1)\, \phi^c \rangle = 2m$$

and

$$\langle q^c, (-\Delta_X^\delta + 1)\, q^c \rangle = 2m$$

Therefore,

$$\exp\{-k[S_X^\delta(q^c)-S_X(\phi^c)]\}=\left[\frac{V_{2m,\delta}(q^c)}{V_{2m}(\phi^c)}\right]^k$$

so we need only estimate the rate of convergence of $V_{2m,\delta}(q^c)$. By defining $\phi_\delta^c(n)\equiv\phi^c(n\delta)$, we obtain the standard Riemann sum convergence estimate

$$|V_{2m,\delta}(\phi_\delta^c)-V_{2m}(\phi^c)|=O(\delta)$$

since ϕ^c is smooth.[37] In order to estimate $V_{2m,\delta}(q^c)-V_{2m,\delta}(\phi_\delta^c)$, we define

$$\psi^c(x)=\phi^c(x)\left[2m\Big/\int_{-T/2}^{T/2}ds\,(\phi^c)^{2m}(s)\right]^{1/(2m-2)}$$

and

$$v_n^c=q_n^c\left[2m\Big/\sum_n\delta(q_n^c)^{2m}\right]^{1/(2m-2)}$$

so that

$$(-\Delta_X+1)\,\psi^c=(\psi^c)^{2m-1}$$

and

$$(-\Delta_X^\delta+1)\,v^c=(v^c)^{2m-1}$$

A standard numerical analysis estimate (Ref. 38, p. 433) then gives us

$$|\psi_\delta^c(n\delta)-v_n^c|=O(\delta^2)$$

which yields

$$\left[\sum_n\delta\,|\psi_\delta^c(n\delta)-v_n^c|^{2m}\right]^{1/2m}=O(\delta^2)$$

This shows that

$$|V_{2m,\delta}(\psi_\delta^c)-V_{2m,\delta}(v^c)|=O(\delta^2)$$

and using

$$\sum_n\delta(\phi_\delta^c)^{2m}(n)=(2m)^m\Big/\left[\sum_n\delta(\psi_\delta^c)^{2m}(n)\right]^{m-1}$$

we obtain

$$|V_{2m,\delta}(\phi_\delta^c)-V_{2m,\delta}(q^c)|=O(\delta^2)$$

Combining our estimates gives us our claim,

$$|S_X^\delta(q^c) - S_X(\phi^c)| = O(\delta)$$

We may now combine

$$\exp[-kS_X^\delta(q^c)] \leqslant \exp[-kS_X(\phi^c)] \exp(bk^{1-\varepsilon})$$

with (4.17) and (4.18) to obtain the upper bound

$$\int V^k(\phi)\, d\mu_X(\phi) \leqslant C_2 \exp(bk^\tau) \exp[-kS_X(\phi)]\, k^{mk} \tag{4.19}$$

This finishes our verification of (e), and so (4.1) now follows from Theorem 2.4.

It is an immediate consequence of (4.19) that

$$\lim_{k\to\infty} \left| \frac{(-1)^k b_k^X(T)}{[(m-1)k]!} \right|^{1/k} \leqslant c_m \exp[-S_X(\phi^c) + m] \tag{4.20}$$

Therefore, (2.3), (2.5a), (4.7), and (4.20) yield

$$\begin{aligned} c_m \exp[(-S_D(\phi^c) + m] \leqslant \lim_{k\to\infty} &\left| \frac{a_k}{[(m-1)k]!} \right|^{1/k} \\ &\leqslant c_m \exp[-S_\mathrm{p}(\phi^c) + m] \end{aligned}$$

The fact that

$$\lim_{T\to\infty} S_X(\phi^c) = \inf_\phi S(\phi), \qquad X = \mathrm{p, D}$$

is easily proven following Ref. 10, Lemma 6.4, and Ref. 11, Lemma 3.1. This completes the proof of (4.2). ▮

4.2. Exponential Potentials

Our potential is assumed to be of the form

$$V(x) = x^m e^{\alpha x}$$

where either

(i) $\alpha > 0, \qquad m = 0, 1, 2, 3,...$

or

(ii) $\alpha \neq 0, \qquad m = 0, 2, 4, 6,...$

We omit the case of $\alpha < 0$ and m odd, since this implies that $V(x)$ is not bounded below. However, we will make some comments on this case later in the section. Exponential potentials have recently been investigated by Maioli[22] and Grecchi *et al.*[23]–[25] In particular, the latter papers are concerned with a generalization of Borel summability to treat perturbation series whose kth perturbation coefficient grows faster than any power of $k!$ for large k. For the above $V(x)$, it is expected that $a_k \sim k^{mk} \exp(c\alpha^2 k^2)/k!$, $c > 0$. When $m = 0$, this was rigorously proven in Ref. 12 using Feynman diagrams. We now extend this result to the $m \neq 0$ cases of (i) and (ii).

Theorem 4.3. Assume that $V(x)$ obeys either condition (i) or (ii). Then there exist positive constants k_0, C_1, C_2, a, b such that for $k \geq k_0$,

$$C_1^k k^{mk} \frac{\exp(a\alpha^2 k^2)}{k!} \leq (-1)^{k+1} a_k \leq C_2^k k^{mk/2} \frac{\exp(b\alpha^2 k^2)}{k!} \tag{4.21}$$

Remark. For $m = 0$, the advantage of the proof of Ref. 12 is that the values for the constants C_1, C_2, a, b are much better. In (4.21) the constants depend on T, and in such a way as to not improve as T increases. The Feynman diagram method has T-independent constants and the values for a and b are very close to the expected exact value (see the remark following Theorem 2.1 of Ref. 12). On the other hand, the Feynman diagram technique does not easily extend to the $m \neq 0$ case.

Proof. It is trivial to verify that conditions (a)–(d) are satisfied by $V(x)$ obeying either (i) or (ii). Therefore, we are left with checking (e′). To do this, it is sufficient to check case (i). If $\alpha > 0$ in (ii), then this is included in (i). Since

$$\int \left[\int_{-T/2}^{T/2} \phi^m(s)\, e^{\alpha\phi(s)}\, ds \right]^k d\mu_X(\phi)$$
$$= (-1)^{mk} \int \left[\int_{-T/2}^{T/2} \phi^m(s)\, e^{-\alpha\phi(s)}\, ds \right]^k d\mu_X(\phi)$$

the $\alpha < 0$ case of (ii) also reduces to (i). However, Theorem 2.3 of Ref. 12 shows that for case (i),

$$C_1^k k^{mk} \frac{\exp(a\alpha^2 k^2)}{k!} \leq (-1)^k b_k^X(T) \leq C_2^k k^{mk/2} \frac{\exp(b\alpha^2 k^2)}{k!} \tag{4.22}$$

for k sufficiently large. This is just (e′) with $g_{1,X}(k) = C_1 k^m \exp(a\alpha^2 k)$ and $g_{2,X}(k) = C_2 k^{m/2} \exp(b\alpha^2 k)$. Briefly, the lower bound of (4.22) is proven by a Jensen's inequality argument if m is even or if m is odd and k is even. If m and k are both odd, then Jensen's inequality is combined with the second GKS inequality to obtain the lower bound. For the upper bound, we use the Schwartz inequality to separate the $\phi^m(s)$ term from the $\exp[\alpha\phi(s)]$ term. The monomial can then be handled as in Section 4.1, while the integral of the exponential term can be evaluated exactly and the result is easy to estimate. Theorem 4.3 is now a consequence of Theorem 2.4. ∎

Another exponential potential that could be handled by our methods is

$$V(x) = x^m \cosh x, \qquad m = 0, 2, 4, 6,...$$

It is straightforward to verify conditions (a)–(e′) for this potential and to see that (4.21) still holds. Also, we could consider

$$V(x) = x^n + x^m e^{\alpha x}$$

with $\alpha > 0$, $m = 0, 1, 2, 3,..., n = 2, 4, 6,....$ The $x^m e^{\alpha x}$ term will yield the dominant contribution to a_k for large k, so that (4.21) is also true for this potential.

Our next example is a potential that oscillates unboundedly.

Theorem 4.4. Assume $V(x) = x^{2m}[1 + \cos(x^{3/2})]$, $m = 0, 1, 2,....$ Then there exist positive constants C_1, C_2, a, b, and k_0 such that for $k \geqslant k_0$,

$$C_1^k k^{(4m-3/2)k} \frac{\exp(ak^4)}{k!} \leqslant (-1)^{k+1} a_k \leqslant C_2^k k^{mk} \frac{\exp(bk^4)}{k!}$$

Remark 1. While V oscillates between zero and infinity as $x \to +\infty$, notice that V increases monotonically to infinity as $x \to -\infty$, since $\cos[(-x)^{3/2}] = \cosh(x^{3/2})$. It is not possible for us to treat an example of a potential that oscillates as x goes to both plus and minus infinity, because the sign condition (d) ensures that V is monotone for at least x positive or x negative.

Remark 2. The correct large-order behavior is presumably

$$|a_k| \sim C^k k^{4mk} \exp(ck^4)$$

Our estimates for this theorem and for Theorem 4.3 are consistent with work of Dolgov and Popov,[35] who formally showed that if

$$V(x) \sim \exp(a\,|x|^{\nu}), \qquad 0 < \nu < 2, \qquad \text{as } |x| \to \infty$$

then

$$|a_k| \sim \exp(bk^\sigma), \qquad k \to \infty$$

with $\sigma = 2/(2-\nu)$. The V of this theorem behaves as $V(x) \sim x^{2m} \exp(|x|^{3/2})$ as $x \to -\infty$.

Remark 3. It is interesting that the coefficients a_k for this potential grow too fast for even the generalized logarithmic Borel summation of Ref. 24 to apply.

Proof. As with the other exponential potentials, the proof reduces to verifying condition (e′). For the upper bound we use that

$$V(x) \leqslant 2x^{2m} \exp(|x|^{3/2})$$

Two applications of the Schwartz inequality then yield

$$\begin{aligned}
&\int V^k(\phi)\, d\mu_X(\phi) \\
&\leqslant 2^k \int \left\{ \int_{-T/2}^{T/2} \phi^{2m}(s) \exp[|\phi(s)|^{3/2}]\, ds \right\}^k d\mu_X(\phi) \\
&\leqslant 2^k \int \left[\int_{-T/2}^{T/2} \phi^{4m}(s)\, ds \right]^{k/2} \left[\int_{-T/2}^{T/2} \exp[2\, |\phi(s)|^{3/2}]\, ds \right]^{k/2} d\mu_X(\phi) \\
&\leqslant 2^k \left\{ \int \left[\int_{-T/2}^{T/2} \phi^{4m}(s)\, ds \right]^k d\mu_X(\phi) \right\}^{1/2} \\
&\quad \times \left[\int \left\{ \int_{-T/2}^{T/2} \exp[2\, |\phi(s)|^{3/2}]\, ds \right\}^k d\mu_X(\phi) \right]^{1/2} \\
&\leqslant C_2^k k^{mk} \left[\int \left\{ \int_{-T/2}^{T/2} \exp[2\, |\phi(s)|^{3/2}]\, ds \right\}^k d\mu_X(\phi) \right]^{1/2}
\end{aligned}$$

where the last line follows from Section 4.1. The inequality

$$\int_{-T/2}^{T/2} \exp[2\, |\phi(s)|^{3/2}]\, ds \leqslant T \exp[2\, \|\phi(s)\|^{3/2}]$$

where $\|\cdot\|$ is sup norm, gives us the bound

$$\int \left\{ \int_{-T/2}^{T/2} \exp[2\, |\phi(s)|^{3/2}]\, ds \right\}^k d\mu_X(\phi) \leqslant T^k \int \exp(2k\, \|\phi\|^{3/2})\, d\mu_X(\phi)$$

The previous integral may be estimated by writing

$$\int \exp(2k\, \|\phi\|^{3/2})\, d\mu_X(\phi) = -\int_0^\infty e^{2kt}\, d\omega(t)$$

in which

$$\begin{aligned}\omega(t) &= \mu_X\{\phi \mid \ \|\phi\|^{3/2} > t\} \\ &= \mu_X\{\phi \mid \ \|\phi\| > t^{2/3}\}\end{aligned}$$

Now, a well-known Gaussian tail estimate (see, for example, Ref. 9, Lemma 18.7) shows that

$$\omega(t) \leqslant C_1 \exp(-C_2 t^{4/3})$$

(this would also follow from the method of large deviations[36]). Therefore, integrating by parts, we obtain

$$\begin{aligned}-\int_0^\infty e^{2kt}\, d\omega(t) &= 1 + 2k \int_0^\infty e^{2kt}\omega(t)\, dt \\ &\leqslant C_1 k \int_0^\infty \exp(2kt - C_2 t^{4/3})\, dt \\ &= C_1 k^4 \int_0^\infty \exp[k^4(2s - C_2 s^{4/3})]\, ds\end{aligned}$$

If we let $f(s) = 2s - C_2 s^{4/3}$, then $f(s)$ has a unique, positive, absolute maximum at $s^* = (3/2C_2)^3$. Since we can assume $k > 1$,

$$k^4 f(s) = (k^4 - 1)\, f(s) + f(s) \leqslant k^4 f(s^*) + f(s)$$

and so

$$\int_0^\infty \exp[k^4 f(s)]\, ds \leqslant \{\exp[k^4 f(s^*)]\} \int_0^\infty [\exp f(s)]\, ds$$

Combining this bound with our previous inequalities yields the upper bound

$$\int V^k(\phi)\, d\mu_X(\phi) \leqslant C_2^k k^{mk} \exp(bk^4)$$

For the lower bound, expanding V in its power series gives

$$\begin{aligned}&\int V^k(\phi)\, d\mu_X(\phi) \\ &\quad = \sum_{n_1=0}^\infty \cdots \sum_{n_k=0}^\infty \alpha_{n_1} \cdots \alpha_{n_k} \int \prod_{i=1}^k \int_{-T/2}^{T/2} ds_i\, \phi^{2m+3n_i}(s_i)\, d\mu_X(\phi)\end{aligned}$$

where $\alpha_0 = 2, \alpha_n = (-1)^n/(2n)!$, $n \geqslant 1$, and the convergence is proven exactly as for the series representations in (3.6) and (3.7). Notice that all terms in this expansion are nonnegative, since $d\mu_X(\phi)$ is even. The lower bound may be found by using the Feynman graph representation

$$\int \prod_{i=1}^{k} \int_{-T/2}^{T/2} ds_i\, \phi^{2m+3n_i}(s_i)\, d\mu_X(\phi)$$
$$= \sum_{\gamma \in \Gamma_k} \int_{-T/2}^{T/2} d^k s \prod_{l \in \gamma} G_X(s_{l_i}, s_{l_f})$$

where now

$$\Gamma_k = \{\text{graphs} \mid k \text{ vertices, vertex } i \text{ has } 2m + 3n_i \text{ lines attached, } i = 1,..., k\}$$

Restricting the s integrations to $[0, 1]^k$ yields

$$\sum_{\gamma \in \Gamma_k} \int_{-T/2}^{T/2} d^k s \prod_{l \in \gamma} G_X(s_{l_i}, s_{l_f}) \geqslant \#(\Gamma_k)[G_X(0, 1)]^{mk + 3(n_1 + \cdots + n_k)/2}$$

The Stirling formula estimates of (3.19) show that

$$\#(\Gamma_k) \geqslant C(2mk + 3n_1 + \cdots + 3n_k)^{mk + 3(n_1 + \cdots + n_k)/2}\, e^{-[mk + 3(n_1 + \cdots + n_k)/2]}$$

Therefore,

$$\sum_{n_1=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} |\alpha_{n_1} \cdots \alpha_{n_k}|\ \#(\Gamma_k)[G_X(0, 1)]^{mk + 3(n_1 + \cdots + n_k)/2}$$
$$\geqslant C \frac{(2mk + 3kn)^{mk + 3kn/2}}{[(2n)!]^k} e^{-mk - 3kn/2}[G_X(0, 1)]^{mk + 3kn/2}$$
$$\geqslant C_1^k (2n)^{-k/2} \frac{(2mk + 3kn)^{mk + 3kn/2}}{(2n)^{2nk}} e^{kn/2}[G_X(0, 1)]^{3kn/2}$$

where we have taken $n_1 = n_2 = \cdots = n_k = n$. The choice $n = c^* k^3$, with $c^* = [G_X(0, 1)\, 3]^3/4^2$, maximizes this lower bound and after some simplification we obtain

$$\int V^k(\phi)\, d\mu_X(\phi) \geqslant C_1^k k^{(4m - 3/2)k} \exp(ak^4)$$

in which $a = c^*/2$. This gives us a lower bound on $\int V^k(\phi)\, d\mu_X(\phi)$ with $g_{1,X}(k) = C_1 k^{(4m - 3/2)} \exp(ak^3)$, which satisfies the required lower bound of condition (e$'$). The theorem now follows from Theorem 2.4. ∎

Finally, we want to mention

$$V(x) = x^m e^{\alpha x}$$

when $\alpha < 0$ and m is odd. While this example is unphysical, in that the resulting operator $H(\lambda)$ is not bounded below, it is interesting to see that a version of the bracketing inequality (2.3) still holds in this case. That is, if we take (2.1) and the second line of (2.2) to be the *definitions*, respectively, of a_k and $a_k^X(T)$, then

$$(-1)^{k+1}\, a_k^{\mathrm{D}}(T) \leqslant (-1)^{k+1}\, a_k \leqslant (-1)^{k+1}\, a_k^{\mathrm{P}}(T), \qquad k \text{ even}$$

$$(-1)^{k+1}\, a_k^{\mathrm{P}}(T) \leqslant (-1)^{k+1}\, a_k \leqslant (-1)^{k+1}\, a_k^{\mathrm{D}}(T), \qquad k \text{ odd}$$

This follows from noticing that when m is odd,

$$\begin{aligned} &u_{k,0}^X(\phi^m(s_1)\exp[\alpha\phi(s_1)],..., \phi^m(s_k)\exp[\alpha\phi(s_k)]) \\ &\quad = (-1)^k\, u_{k,0}^X(\phi^m(s_1)\exp[-\alpha\phi(s_1)],..., \phi^m(s_k)\exp[-\alpha\phi(s_k)]) \end{aligned}$$

so the $\alpha < 0$ case is related to the $\alpha > 0$ case, for which we have already proven (2.3). Since for this $V(x)$, $(-1)^{k+1}\, a_k$ and $(-1)^{k+1}\, a_k^X(T)$ are nonnegative for k even and nonpositive for k odd, we can combine the above into a single bracketing inequality

$$|a_k^{\mathrm{D}}(T)| \leqslant |a_k| \leqslant |a_k^{\mathrm{P}}(T)|$$

The main reason that we have mentioned the $\alpha < 0$, m odd case of the exponential potential is that it gives us another example where the sufficient condition (d) is violated, and yet the bracketing inequality (2.3) (or its reverse when k is odd) remains true.

4.3. $P(\phi)_2$ Euclidean Quantum Fields

Our last example is that of a polynomial interaction for a two-dimensional continuum Euclidean quantum field theory. We will also briefly mention d-dimensional lattice Euclidean fields. Our basic result is to extend the results of Theorem 4.1 to these examples. For background on Euclidean field theory see Refs. 26–31. To begin, define the partition function $Z_X(\lambda)$ as

$$Z_X(\lambda) = \int e^{-\lambda V(\phi)}\, d\mu_\Lambda^X$$

where $\Lambda = [-T/2, T/2]^2$, and $d\mu_\Lambda^X$ is the mean-zero Gaussian measure with covariance

$$\int \phi(x)\, \phi(y)\, d\mu_\Lambda^X = (-\Delta_X + 1)^{-1}(x, y)$$

where Δ_X is the Laplacian on Λ obeying $X=$ p, D, 0 boundary conditions. The interaction $V(\phi)$ is

$$V(\phi)=\int_\Lambda :P(\phi)(x):\, d^2x$$

in which

$$P(x)=\sum_{n=0}^{2m} \alpha_n x^n$$

with either

(i) $\alpha_n \geqslant 0, \quad \forall n$

or

(ii) $(-1)^n \alpha_n \geqslant 0, \quad \forall n$

and we assume $\alpha_{2m}=1$ for convenience. The Wick ordering is formally defined by

$$:\phi^n(x):=\sum_{j=0}^{[n/2]} \frac{(-1)^j n!}{j!\,(n-2j)!\,2^j} G_X(x,x)^j \phi^{n-2j}(x)$$

Since all terms in this expression are infinite, $V(\phi)$ must first be regularized and then defined in the limit as the regularization is removed.[26,27] The pressure is defined as

$$p(\lambda)=\lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|} \ln Z_X(\lambda)$$

Existence of the limit is shown by either a cluster expansion[26,29] for small $\lambda \geqslant 0$, or by conditioning and decoupling for all $\lambda \geqslant 0$.[26,27,30,31] That the pressure is independent of the boundary conditions $X=$ p, D, 0 is also a result of Ref. 31. The cluster expansion results show that $p(\lambda)$ has an asymptotic perturbation series[32]

$$p(\lambda) \sim \sum_{k=0}^{\infty} a_k \lambda^k$$

as $\lambda \to 0^+$. As in Section 4.1, we define functionals

$$S(\phi)=\frac{1}{2}\int_{R^2} [(\nabla\phi)^2(x)+\phi^2(x)]\, d^2x - \ln \int_{R^2} \phi^{2m}(x)\, d^2x$$

and

$$S_X(\phi)=\frac{1}{2}\int_\Lambda [(\nabla_X\phi)^2(x)+\phi^2(x)]\,d^2x$$
$$-\ln\int_\Lambda \phi^{2m}(x)\,d^2x, \qquad X=\mathrm{p, D}$$

which are defined on appropriate Sobolev spaces. These functionals are bounded below and attain their infimums (for details, see Ref. 11). Our asymptotic result is the following.

Theorem 4.5. Let $P(x)$ obey either (i) or (ii). Then there exist positive constants k_0, C_1, C_2, D_1, D_2, a, b, γ, τ, with $\gamma, \tau < 1$, such that for all $k \geqslant k_0$,

$$C_1\exp(ak^\gamma)\,D_1^k\frac{k^{mk}}{k!}\leqslant(-1)^k a_k\leqslant C_2\exp(bk^\tau)\,D_2^k\frac{k^{mk}}{k!}$$

Furthermore,

$$\lim_{k\to\infty}\left|\frac{a_k}{[(m-1)k]!}\right|^{1/k}=c_m\exp[-\inf S(\phi)+m] \tag{4.23}$$

Remark. For $P(x)=x^4$, (4.23) was first proven in Ref. 11. The constants D_1, D_2 are exactly the two-dimensional versions of the expressions for D_1, D_2 given in Remark 1 following Theorem 4.1.

Only the dimension has changed in going from Theorem 4.1 to Theorem 4.5. The path integral framework shows the unity of the examples of Sections 4.1 and 4.3, since we may view them together as $P(\phi)_d$ Euclidean fields for $d=1, 2$. The proof follows a pattern very similar to Section 4.1, so we will be brief. Details can be filled in from the proof of Theorem 4.1 and from Ref. 11. We use

$$a_k^X(\Lambda)\equiv\frac{1}{|\Lambda|\,k!}\frac{d^k}{d\lambda^k}\bigg|_{\lambda=0}\ln Z_X(\lambda)$$
$$=\frac{(-1)^k}{k!}\int_\Lambda d^{2k}x\,u_{k,0}^0(:P(\phi)(x_1):,..., :P(\phi)(x_k):), \qquad X=\mathrm{p, D, 0}$$

to obtain a bracketing inequality, where $u_{k,0}^X$ is the kth Ursell function with respect to the measure $d\mu_\Lambda^X$.

Lemma 4.6. For all $k\geqslant 1$ and T,

$$(-1)^k a_k^{\mathrm{D}}(\Lambda)\leqslant(-1)^k a_k\leqslant(-1)^k a_k^{\mathrm{p}}(T) \tag{4.24}$$

A version of Theorem 2.3 holds for this example, since

$$Z_X(\lambda) \sim \sum_{k=0}^{\infty} b_k^X(\Lambda)\,\lambda^k$$

with

$$b_k^X(\Lambda) = \frac{(-1)^k}{k!} \int V^k(\phi)\, d\mu_\Lambda^X$$

Lemma 4.7. For all k sufficiently large and for all T,

$$a_k^X(\Lambda) = (1/|\Lambda|)\, b_k^X(\Lambda)[1 - O(1/k^{m-1})] \tag{4.25}$$

The last results we need are the upper and lower bounds on $b_k^X(\Lambda)$.

Theorem 4.8. Let $X = \mathrm{p}, \mathrm{D}$. Then for $k \geqslant k_0$

$$C_1 \exp(ak^\gamma) \exp[-kS_X(\phi^c)]\, k^{mk}$$
$$\leqslant \int V^k(\phi)\, d\mu_\Lambda^X \leqslant C_2 \exp(bk^\tau) \exp[-kS_X(\phi^c)]\, k^{mk}$$

and so

$$\lim_{k\to\infty} k^{-m} \left[\int V^k(\phi)\, d\mu_\Lambda^X\right]^{1/k} = \exp[-S_X(\phi^c)] \tag{4.26}$$

Remark. In (4.26), ϕ^c is the function that minimizes $S_X(\phi)$. All comments made about the ϕ^c of Section 4.1 at the beginning of the proof of Theorem 4.1 apply in this case also.

These three results are all the input needed for proving Theorem 4.5.

Proof of Theorem 4.5. Combining (4.24), (4.25), and the upper and lower bounds of Theorem 4.8 yields the claimed upper and lower bounds of Theorem 4.5. To prove (4.23), we combine (4.24)–(4.26) to obtain

$$c_m \exp[-S_{\mathrm{D}}(\phi^c) + m] \leqslant \lim_{k\to\infty} \left|\frac{a_k}{[(m-1)\,k]!}\right|^{1/k}$$
$$\leqslant c_m \exp[-S_{\mathrm{p}}(\phi^c) + m]$$

The limit in (4.23) then follows from

$$\lim_{|\Lambda|\to\infty} S_X(\phi^c) = \inf S(\phi), \qquad X = \mathrm{p}, \mathrm{D}$$

which is proven exactly as in Ref. 11, Lemma 3.1. ∎

Proof of Lemma 4.6. From Ref. 32, we know that

$$a_k = \frac{(-1)^k}{k!} \int_{R^{2k-2}} d^{2k-2}x \, u^0_{k,0}(:P(\phi)(0):, :P(\phi)(x_2): ,..., :P(\phi)(x_k):)$$

and it is easy to see that

$$a^X_k(\Lambda) = \frac{(-1)^k}{|\Lambda| \, k!} \int_\Lambda d^{2k}x \, u^X_{k,0}(:P(\phi)(x_1):,..., :P(\phi)(x_k):)$$

These may be rewritten as

$$a_k = \frac{(-1)^k}{k!} \sum_{n_1=0}^{2m} \cdots \sum_{n_k=0}^{2m} \alpha_{n_1} \cdots \alpha_{n_k} \times \int_{R^{2k-2}} d^{2k-2}x \, u^0_{k,0}(:\phi^{n_1}(0):, :\phi^{n_2}(x_2):,..., :\phi^{n_k}(x_k):) \qquad (4.27)$$

and

$$a^X_k(\Lambda) = \frac{(-1)^k}{|\Lambda| \, k!} \sum_{n_1=0}^{2m} \cdots \sum_{n_k=0}^{2m} \alpha_{n_1} \cdots \alpha_{n_k} \times \int_\Lambda d^{2k}x \, u^X_{k,0}(:\phi^{n_1}(x_1):,..., :\phi^{n_k}(x_k):) \qquad (4.28)$$

The Ursell functions have a Feynman diagram representation as

$$\int_{R^{2k-2}} d^{2k-2}x \, u^0_{k,0}(:\phi^{n_1}(0):, :\phi^{n_2}(x_2):,..., :\phi^{n_k}(x_k):) = \sum_{\gamma \in \Gamma_k} \int_\Lambda d^{2k-2}x \prod_{l \in \gamma} G_0(x_{l_i} - x_{l_f}) \qquad (4.29)$$

and

$$\int_\Lambda d^{2k}x \, u^X_{k,0}(:\phi^{n_1}(x_1):,..., :\phi^{n_k}(x_k):) = \sum_{\gamma \in \Gamma_k} \int_\Lambda d^{2k}x \prod_{l \in \gamma} G_X(x_{l_i}, x_{l_f}) \qquad (4.30)$$

where

$$\Gamma_k = \{\text{connected graphs} \mid k \text{ vertices, no self-loops, vertex } i \text{ has } n_i \text{ lines attached, } i = 1,..., k\}$$

From the Feynman diagram representation and our restriction on the signs of the α_n, it follows that

$$\alpha_{n_1} \cdots \alpha_{n_k} \int_\Lambda d^{2k}x \, u^X_{k,0}(:\phi^{n_1}(x_1):,..., :\phi^{n_k}(x_k):) \geqslant 0$$

and similarly for (4.27). This uses the fact that $u^X_{k,0} = 0$ if $n_1 + \cdots + n_k$ is odd. Combining (4.27)–(4.30), we see that a_k and $a^X_k(\Lambda)$ have expressions just like (3.6)–(3.10) of the proof of Theorem 2.2. The only difference is that the Wick-ordering suppresses graphs with self-loops. Therefore, the proof goes through exactly as before. In fact, it is easier, since the sums in (4.27), (4.28) are finite. ∎

We omit the proof of Lemma 4.7, since it is virtually indentical to the proof of Theorem 2.3. Our upper and lower bounds from Theorem 4.8 are of exactly the same form as those found in the proof of Theorem 4.1. Therefore, the proof of Theorem 2.3 for assumption (e) will work for Lemma 4.7. We wish to point out that the proof of Theorem 2.3 for assumption (e) corrects an error in the proof of Lemma 3.1 in Ref. 11. The earlier proof relied on the assumption that

$$\frac{\int |V(\phi)|^k \, d\mu}{k!}$$

was log convex in k. While we still believe this to be true, the proof of log convexity given in Ref. 11 is incorrect.

Proof of Theorem 4.8. The proof is a combination of the proof of Theorem 4.1 and of the proof of Theorem 1.2 of Ref. 11. Our Jensen's inequality argument for the lower bound goes through unchanged for the case of k even. For the upper bound, it is necessary to use the lattice approximation, as we did in the upper bound of Theorem 4.1. The estimates require that we now choose $\varepsilon < 1/2$ in $\delta = T/(2[k^\varepsilon] + 1)$ in order to control remainder terms. For the lower bound when k is odd, we must also use the lattice approximation with our k-dependent choice of δ. This is because of the infinite counterterms in the Wick ordering. The difference between the continuum and lattice theory is then controlled by using Lemma 2.2 of Ref. 11. This may be easily integrated into the lower bound proof in Theorem 4.1. ∎

In Section 4.1 we briefly mentioned the anharmonic oscillator with a Wick-ordered polynomial for a potential. If we consider

$$H(\lambda) = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2 - 1 \right) + \lambda :P(x):$$

where $P(x)$ obeys either (i) or (ii) of Section 4.3, then the bracketing inequality (2.3) holds, by the proof of Lemma 4.6, and consequently (4.2) holds for this choice of potential. In this case, there are no infinities in $:P(x):$, since $G_X(x, x)$ is finite for dimension $d=1$. Therefore, if $P(x)$ is a semibounded polynomial obeying our sign convention, then $:P(x):$ is simply a new semibounded polynomial, but for which the sign convention is no longer true. This gives us an example of an interesting class of potentials for which condition (d) does not hold, and yet all of our results, including the bracketing inequality, are true.

If $S^\delta(q)$ is the d-dimensional lattice version of the functional $S(\phi)$, then the proof of Theorem 4.5 easily extends to show the following (we state only the result for the $|a_k|^{1/k}$ asymptotics).

Theorem 4.9. Let a_k be the kth perturbation coefficient for the pressure in a d-dimensional lattice Euclidean field theory with interaction $P(x)$ obeying either (i) or (ii). Then

$$\lim_{k\to\infty} \left| \frac{a_k}{[(m-1)k]!} \right|^{1/k} = c_m \exp[-\inf S^\delta(q) + m]$$

Remark. This was first proven for $P(x)=x^4$ in Ref. 13.

Finally, we want to mention that for the case of $P(x)=x^4$, Magnen and Rivasseau[8] have extended (4.23) to three dimensions. This requires a combination of the proof of Theorem 4.5 with an interesting Feynman diagram analysis necessary to deal with the mass counterterm of $(\phi^4)_3$.

5. DISCUSSION

There are three problems we want to mention as possible areas for further work on large order (see also Ref. 1, Section 7 for a list of specific problems). The first is that of obtaining large-order estimates for higher eigenvalues. If

$$E_i(\lambda) \sim \sum_{k=0}^{\infty} a_k^i \lambda^k$$

as $\lambda \to 0^+$, where $E_i(\lambda)$ is the ith eigenvalue of $H(\lambda)$, then the dominant term in the large-order behavior of a_k^i should be independent of i. In particular, for $V(x)=x^{2m}$, $a_k^i \sim [(m-1)k]!$, $\forall i$ (see Ref. 2 for a statement of the detailed asymptotics), and for $V(x)=\exp(\alpha x)$, $a_k^i \sim \exp(\alpha^2 k^2/4)/k!$, $\forall i$.[25] Since path integral methods work so well for the lowest eigenvalue, it would be interesting to have a result that said that large-order estimates for the lowest eigenvalue implied large-order estimates for the higher eigenvalues. Of course, this is not a problem for WKB methods, since they work for all eigenvalues. However, WKB has so far been used to obtain very detailed asymptotics for particular V's.[4,5] There is no proof by WKB methods of large-order estimates for a general class of V's in the spirit of Theorem 2.4. In particular, for the potential $V(x) = x^m \exp(\alpha x)$, treated in Section 4.2, path integrals (and Feynman diagrams for $m = 0$) provide the only proof so far of divergence of the perturbation series and of large-order estimates for the lowest eigenvalue. Therefore, it would be worthwhile to be able to extend this result to higher eigenvalues.

It would also be interesting to know when either Theorem 2.2 or Theorem 2.4 can be obtained under a weakening of assumptions (a)–(e) or (e′). As explained previously, (b) and (c) are quite reasonable, while (d) seems the obvious condition to try to relax. In Section 4.1 and 4.3, examples were given of potentials V for which (d) was false, but Theorems 2.2 and 2.4 were true. A particular case worth doing would be to prove (4.2) or (4.23) with

$$V(x) \equiv P(x) = \sum_{n=0}^{2m} \alpha_n x^n$$

$\alpha_{2m} = 1$, and no restriction on the signs of α_n, $n = 0,..., 2m-1$. In regards to weakening either (a) or (d), it would be helpful to know how the signs of the Ursell functions in (2.1) and (2.2) vary with k for more general V's. For V's obeying (a)–(d), the Ursell functions were nonnegative, since we could expand the V's in power series, explicitly evaluate the Ursell functions of monomials, and then use our sign condition (d). The sign dependence is important for the current proof of Theorem 2.2.

For (e) and (e′), it would be helpful to weaken the required estimate on $f^j_{2,X}(j)/f^{j-1}_{1,X}(k-1)$ in (e) and the lower bound on $g_{1,X}(k)$ in (e′). The estimate in (e) means that $f_{1,X}$ and $f_{2,X}$ have to be reasonably close to each other. It is also implicit in this estimate that $f_{1,X}$ is growing at least as fast as k, which seems reasonable, since all the examples in Section 4 grow at least that fast. For most of the proof of Theorem 2.3, only the lower bounds on $\int V^k(\phi)\, d\mu_X(\phi)$ in (e) and (e′) are needed. The additional estimates in (e) and (e′) are both needed only to handle the term

$$\sum_{j=k_0}^{k-k_0} |b_j b_{k-j}/b_{k-1}|$$

used in proving (3.24). An alternate approach to this term would be to assume that $|b_k|$ is log convex in k, since then

$$|b_j b_{k-j}| \leqslant |b_{k_0} b_{k-k_0}|, \qquad k_0 \leqslant j \leqslant k - k_0$$

However, it has not been possible for us to prove log convexity of $|b_k|$ for any of the examples we treat in Section 4. While it seems reasonable to believe that log convexity is true for the examples, it appears to be difficult to prove. Our present assumptions are capable of handling a potential that grows polynomially [condition (e)] or like $\exp(|x|^\nu)$ for $1 \leqslant \nu < 2$ [condition (e′)]. We will not be able to prove Theorem 2.3 for a potential that grows faster than a polynomial but slower than $\exp(|x|)$. An example would be $V(x) = x^{2m}[1 + \cos(x^{1/2})]$. Since $V(x) \sim x^{2m} \exp(|x|^{1/2})$ as $x \to -\infty$, we would expect that $|a_k| \sim \exp(ak^{4/3})$ as $k \to \infty$, by Remark 2 following Theorem 4.4. This is not fast enough for the lower bound on $g_{1,x}$ in (e′) to hold. While (e) could apply in principle, if we use the methods of Theorem 4.4 on this potential we will obtain upper and lower bounds with dominant term $\exp(ak^{4/3})$, but with different values of a in the upper and lower bounds. Therefore, we will be unable to verify the estimate on $f^j_{2,x}(j)/f^{j-1}_{1,x}(k-1)$.

Perhaps the most interesting (and difficult) extension of our results would be to potentials $V(x, \lambda)$ that are nonlinear in λ. The best known example is the double-well anharmonic oscillator,

$$H(\lambda) = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2 - 1\right) + \lambda x^3 + \frac{\lambda^2}{2} x^4$$

for which the ground-state energy perturbation coefficients are expected to behave like

$$a_{2k} \sim -\frac{3}{\pi} 3^k k!$$

for large k ($a_{2k+1} = 0, \forall k$). This is based on numerical calculations[18,33,34] and an interesting but formal path integral argument (Lipatov method).[18] Proving upper and lower bounds that grow like $k!$ for the a_{2k} of this example would be a good place to start for nonlinear potentials. Finding methods that yield large-order estimates for more general nonlinear V should be subtle. As mentioned in Section 4.1, lower order terms in nonlinear V can very dramatically affect the large-order behavior of the perturbation coefficients. For example, simply changing the double-well Hamiltonian to

$$H(\lambda) = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2 - 1\right) + \lambda(x^3 - x) + \frac{\lambda^2}{2} x^4$$

produces a ground-state energy with perturbation coefficients $a_k = 0, \forall k$ (see Ref. 21)!

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